Exemple 1

$x_1, \dots, x_n \sim \mathcal{N}(\mu, \sigma^2)$

$\underline{x} = (x_1, \dots, x_n)$; $\underline{x} \mid \mu, \sigma^2 \sim \mathcal{N}^{\otimes n}(\mu, \sigma^2)$

$\mu \in \mathbb{R}$, $\sigma^2 > 0$ inconnus

On se place dans le contexte bayésien.

$\pi(\mu, \sigma^2) = \pi(\mu)\pi(\sigma^2)$ a priori $\mu \perp\!\!\!\perp \sigma^2$

$\mu \sim \mathcal{U}_{[-3,3]}$, $\sigma^2 \sim \mathcal{IG}(1,1)$

Target : $\pi(\mu, \sigma^2 \mid \underline{x})$ loi a posteriori

$$\pi(\mu, \sigma^2 \mid \underline{x}) \propto f(\underline{x} \mid \mu, \sigma^2)\, \pi(\mu, \sigma^2)$$

$$\pi(\mu, \sigma^2 \mid \underline{x}) \propto \ell(\mu, \sigma^2 \mid \underline{x})\, \pi(\mu)\pi(\sigma^2)$$

$$\pi(\mu, \sigma^2 \mid \underline{x}) \propto \boxed{e^{-\frac{1}{2\sigma^2}\sum (x_i - \mu)^2} \left(\frac{1}{\sigma}\right) \mathbb{1}_{[-3,3]}(\mu) \left(\frac{1}{\sigma^2}\right)^2 e^{-\left(\frac{1}{\sigma^2}\right)} \mathbb{1}_{\{\sigma^2 > 0\}}}$$

MH : Proposition pour (μ, σ^2), pourquoi pas une marche gaussienne mais alors comment choisir la matrice de covariance $\Longrightarrow$ difficile

Gibbs sampler

$$\pi(\mu \mid \sigma^2, \underline{x}) = \frac{f(\underline{x} \mid \mu, \sigma^2)\,\pi(\mu, \sigma^2)}{\int f(\underline{x} \mid \mu, \sigma^2)\,\pi(\mu, \sigma^2)\,d\mu}$$

$$(\mathbb{P}(A \mid B) = \mathbb{P}(A \cap B) / \mathbb{P}(B))$$

$$\pi(\mu \mid \sigma^2, \underline{x}) \propto f(\underline{x} \mid \mu, \sigma^2)\,\pi(\mu, \sigma^2)$$

$$\pi(\mu \mid \sigma^2, \underline{x}) \propto \ell(\mu, \sigma^2 \mid \underline{x})\,\pi(\mu)\,\cancel{\pi(\sigma^2)}$$

$$\pi(\mu \mid \sigma^2, \underline{x}) \propto e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}\left(\frac{1}{\sigma}\right)\mathbb{I}_{[-3,3]}(\mu)$$

$$\pi(\mu \mid \sigma^2, \underline{x}) \propto e^{-\frac{1}{2\sigma^2}(-2\mu\sum x_i + n\mu^2)}\,\mathbb{I}_{[-3,3]}(\mu)$$

$$\pi(\mu \mid \sigma^2, \underline{x}) \propto e^{-\frac{n}{2\sigma^2}(\mu^2 - 2\mu\bar{x})}\,\mathbb{I}_{[-3,3]}(\mu)$$

$$\pi(\mu \mid \sigma^2, \underline{x}) \propto e^{-\frac{n}{2\sigma^2}(\mu - \bar{x})^2}\,\mathbb{I}_{[-3,3]}(\mu)$$

$$\Rightarrow \mu \mid \sigma^2, \underline{x} \sim N_{[-3,3]}\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

$$\pi(\sigma^2 \mid \mu, \underline{x}) = \frac{f(\underline{x} \mid \mu, \sigma^2)\,\pi(\mu, \sigma^2)}{\int f(\underline{x} \mid \mu, \sigma^2)\,\pi(\mu, \sigma^2)\,d\sigma^2}$$

$$\pi(\sigma^2 \mid \mu, \underline{x}) \propto f(\underline{x} \mid \mu, \sigma^2)\,\pi(\mu, \sigma^2)$$

$$\pi(\sigma^2 \mid \mu, \underline{x}) \propto \ell(\mu, \sigma^2 \mid \underline{x})\,\pi(\sigma^2)\,\cancel{\pi(\mu)}$$

$$\pi(\sigma^2 \mid \mu, \underline{x}) \propto e^{-\frac{1}{2\sigma^2}\sum (x_i-\mu)^2} \left(\frac{1}{\sigma^2}\right)^2 e^{-\frac{1}{\sigma^2}} \mathbb{1}_{\{\sigma^2>0\}}$$

$$\pi(\sigma^2 \mid \mu, \underline{x}) \propto \exp\left[-\frac{1}{\sigma^2}\left[\frac{1}{2}\sum (x_i-\mu)^2 + 1\right]\right] (\sigma^2)^{-\frac{n}{2}} \left(\frac{1}{\sigma^2}\right)^2 \mathbb{1}\{\sigma^2>0\}$$

$$\Rightarrow \sigma^2 \mid \mu, \underline{x} \sim \mathcal{IG}\left(\frac{n}{2}+1, \frac{\sum (x_i-\mu)^2}{2}+1\right)$$

On part de $(\sigma^2)^{(0)}$

On simule $\mu^{(1)} \sim \mathcal{N}_{[-3,3]}\left(\bar{x}, \frac{(\sigma^2)^{(0)}}{n}\right)$

On simule $(\sigma^2)^{(1)} \sim \mathcal{IG}\left(\frac{n}{2}+1, \frac{\sum (x_i-\mu^{(1)})^2}{2}+1\right)$

On simule $\mu^{(2)} \sim \mathcal{N}_{[-3,3]}\left(\bar{x}, \frac{(\sigma^2)^{(1)}}{n}\right)$

On simule $(\sigma^2)^{(2)} \sim \mathcal{IG}\left(\frac{n}{2}+1, \frac{\sum (x_i-\mu^{(2)})^2}{2}+1\right)$

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Exemple 2

$x_1, \ldots, x_n \sim \mathcal{N}(\mu, \sigma^2)$

$\mu \sim \mathcal{U}_{[-3,3]}$ $\quad \mu \perp\!\!\!\perp \sigma^2$ a priori

$\sigma^2 \sim \mathcal{U}_{[1,5]}$

$\mu \mid \sigma^2, \underline{x} \sim \mathcal{N}_{[-3,3]}\left(\bar{x}, \frac{\sigma^2}{n}\right)$

$\pi(\sigma^2 \mid \mu, \underline{x}) \propto e^{-\frac{1}{2\sigma^2}\sum (x_i-\mu)^2} (\sigma^2)^{-\frac{n}{2}} \mathbb{1}_{[1,5]}(\sigma^2)$

Étape MH dans Gibbs pour simuler suivant $\pi(\sigma^2 \mid \mu, \underline{x})$

Par exemple en utilisant un noyau de proposition indépendant suivant une loi uniforme sur $[1,5]$!

Cible de MH c'est $\pi(\sigma^2 \mid \mu, \underline{x})$!