Ex. 1

$f(x|\theta) = C_n^x \theta^x (1-\theta)^{n-x}$ | $x \in \{0, ----, n\}$

$x \in \{0, ----, n\}$ | $\ell(\theta|x) = \boxed{C_n^x} \theta^x (1-\theta)^{n-x}$

θ fixé | x fixé (observé)

x inconnue | θ inconnue

$\pi(\theta) \longrightarrow \pi(\theta|x) \longrightarrow \hat{\theta}$

1) $x \in \{0, ----, n\}$

$$\log(\ell(\theta|x)) = \log(C_n^x) + x\log(\theta) + (n-x)\log(1-\theta)$$

$$\frac{d\log\ell}{d\theta}(\theta|x) = \left(\frac{x}{\theta}\right) - \left(\frac{n-x}{1-\theta}\right)$$

$$\frac{d^2\log\ell}{(d\theta)^2}(\theta|x) = -\frac{x}{\theta^2} - \frac{(n-x)}{(1-\theta)^2}$$

$$I_x(\theta) = -\mathbb{E}\left[-\frac{X}{\theta^2} - \frac{(n-X)}{(1-\theta)^2}\right]$$

$$I_x(\theta) = \frac{n}{\theta} + \frac{n}{(1-\theta)} = \frac{n}{\theta(1-\theta)}$$

$$\pi_1(\theta) \propto \sqrt{|I_1(\theta)|}\, \mathbb{1}_{[0,1]}(\theta)$$

$$\pi_1(\theta) \propto \underbrace{\theta^{-1/2}(1-\theta)^{-1/2}}\, \mathbb{1}_{[0,1]}(\theta)$$

$\theta \sim \mathrm{Beta}\left(\frac{1}{2}, \frac{1}{2}\right)$

loi a priori de Jeffreys

$$\pi_1(\theta|x) \propto \ell(\theta|x)\,\pi_1(\theta)$$

$$\pi_1(\theta|x) \propto \theta^{x}(1-\theta)^{n-x}\,\theta^{-1/2}(1-\theta)^{-1/2}\,\mathbb{1}_{[0,1]}(\theta)$$

$$\pi_1(\theta|x) \propto \theta^{x-1/2}(1-\theta)^{n-x-1/2}\,\mathbb{1}_{[0,1]}(\theta)$$

$$\boxed{\theta|x \sim \mathrm{Beta}\left(x+\frac{1}{2},\, n-x+\frac{1}{2}\right)}$$

Estimateur bayésien

Classiquement l'espérance de la loi a posteriori

$$\mathbb{E}(\theta|x) = \frac{x+1/2}{x+1/2+n-x+1/2}$$

θ_1 EMV

$$\mathbb{E}(\theta|x) = (x+1/2)/(n+1) \neq \frac{x}{n}$$

2) $\pi_2(\theta|x) \propto \ell(\theta|x)\, \pi_2(\theta)$

$\Rightarrow \pi_2(\theta|x) \propto \theta^{x}(1-\theta)^{n-x}\, \mathbb{1}_{[0,1]}(\theta)$

$\Rightarrow \theta|x \sim Beta(x+1, n-x+1)$

$\Rightarrow \hat{\theta}_2 = \left[\dfrac{x+1}{x+1+n-x+1}\right] = \left[\dfrac{x+1}{n+2}\right]$

$\neq \left[\hat{\theta}_1 = \left[\dfrac{x+1/2}{n+1}\right]\right] \neq \left[\hat{\theta}^{MV} = \dfrac{x}{n}\right]$

Ex. 3

$\underline{x} = (x_1, ----, x_n)$

$\forall i \in \{1, ----, n\} \quad x_i \in \mathbb{R}_+^*$

$\ell(\theta|\underline{x}) \propto \prod_{i=1}^{n} \exp\left[-\frac{1}{2}\left[\log(x_i) - \theta\right]^2\right]$

$\ell(\theta|\underline{x}) \propto \left\{\exp\left[-\frac{1}{2}\sum_{i=1}^{n}(\log(x_i) - \theta)^2\right]\right\}$

$\pi(\theta) \propto \exp\left(-\dfrac{\theta^2}{2}\right)$

$\pi(\theta|\underline{x}) \propto \boxed{\ell(\theta|x)\, \pi(\theta)}$

$\pi(\theta|\underline{x}) \propto \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(\log(x_i) - \theta)^2 - \dfrac{\theta^2}{2}\right]$

$\pi(\theta|\underline{x}) \propto \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\theta^2 - 2\theta\log(x_i)\right) - \dfrac{\theta^2}{2}\right]$

$$\pi(\theta|\underline{x}) \propto \exp\left[-\frac{1}{2}\left((n+1)\theta^2 - 2\theta\sum_{i=1}^{n}\log(x_i)\right)\right]$$

$$\pi(\theta|\underline{x}) \propto \exp\left[-\left(\frac{n+1}{2}\right)\left(\theta^2 - \frac{2}{n+1}\theta\sum_{i=1}^{n}\log(x_i)\right)\right]$$

$$\pi(\theta|\underline{x}) \propto \exp\left[-\left(\frac{n+1}{2}\right)\left(\theta - \frac{\sum_{i=1}^{n}\log(x_i)}{n+1}\right)^2\right]$$

$$\Rightarrow \theta|\underline{x} \sim \mathcal{N}\left(\frac{\sum_{i=1}^{n}\log(x_i)}{n+1}, \frac{1}{n+1}\right)$$

$$\Rightarrow \hat{\theta} = \left(\frac{\sum_{i=1}^{n}\log(x_i)}{n+1}\right)$$

$$L(\theta, u) = (\theta - u)^2$$

$$\mathbb{E}_{\theta|\underline{x}}\left((\theta - u)^2\right)$$

$$= \mathbb{E}_{\theta|\underline{x}}\left[\theta^2 - 2\theta u + u^2\right]$$

$$= \underbrace{\mathbb{E}_{\theta|\underline{x}}(\theta^2) - 2u\,\mathbb{E}_{\theta|\underline{x}}(\theta) + u^2}_{h(u)}$$

On cherche u^* qui minimise $h(u)$

$$h(d) = \mathbb{E}_{\theta|\underline{x}}(\theta^2) - 2d\,\mathbb{E}_{\theta|x}(\theta) + d^2$$

$$h''(d) = -2\,\mathbb{E}_{\theta|\underline{x}}(\theta) + 2d$$

≥ 0 $h(\cdot)$ strettamente convessa

d^* t.q. : $-2\,\mathbb{E}_{\theta|\underline{x}}(\theta) + 2d^* = 0$

$\iff d^* = \mathbb{E}_{\theta|x}(\theta)$

Speranza della legge a posteriori

$$L(\theta, d) = |\theta - d|$$

d^* mediana della legge a posteriori

Ex. 3

$\theta \in \{0, 2\}$ $\quad \mathbb{P}(\theta = 0) = \mathbb{P}(\theta = 2) = \frac{1}{2}$

$\underline{Z} = (Z_1, Z_2) \mid \begin{array}{l} Z_1 \mid \theta \sim N(\theta, 1) \\ Z_2 \mid \theta \sim N(\theta, 1) \end{array} \mid \underline{Z_1 \perp\!\!\!\perp Z_2 \mid \theta}$

$$\ell(\theta \mid \underline{Z}) \propto \left[\exp\left(-\frac{1}{2}(Z_1 - \theta)^2\right) \exp\left(-\frac{1}{2}(Z_2 - \theta)^2\right)\right]$$

$$\left[\begin{array}{l} \mathbb{P}(\theta = 0 \mid \underline{Z}) \propto \ell(0 \mid \underline{Z})\, \mathbb{P}(\theta = 0) \\ \mathbb{P}(\theta = 2 \mid \underline{Z}) \propto \ell(2 \mid \underline{Z})\, \mathbb{P}(\theta = 2) \end{array}\right]$$

$$\mathbb{P}(\theta = 0 \mid Z) \propto e^{-\frac{Z_1^2}{2} - \frac{Z_2^2}{2}} \left(\frac{1}{2}\right)$$

$$\mathbb{P}(\theta = 2 \mid Z) \propto e^{-\frac{(Z_1 - 2)^2}{2} - \frac{(Z_2 - 2)^2}{2}} \left(\frac{1}{2}\right)$$

$$\Rightarrow \mathbb{P}(\theta = 0 \mid \underline{Z}) = \frac{e^{-Z_1^2/2 - Z_2^2/2} \cancel{\left(\frac{1}{2}\right)}}{\cancel{\left(\frac{1}{2}\right)} e^{-Z_1^2/2 - Z_2^2/2} + e^{-\frac{(Z_1-2)^2}{2} - \frac{(Z_2-2)^2}{2}} \cancel{\left(\frac{1}{2}\right)}}$$

$$\Rightarrow \mathbb{P}(\theta = 2 \mid \underline{Z}) = \frac{\cancel{\left(\frac{1}{2}\right)} e^{-(Z_1-2)^2/2 - (Z_2-2)^2/2}}{\cancel{\left(\frac{1}{2}\right)} e^{-Z_1^2/2 - Z_2^2/2} + e^{-(Z_1-2)^2/2 - (Z_2-2)^2/2} \cancel{\left(\frac{1}{2}\right)}}$$

$$\mathbb{P}(\theta = 0 \mid Z) + \mathbb{P}(\theta = 2 \mid \underline{Z}) = 1$$