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Optimal joint replenishment decisions for a central factory with multiple satellite factories

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Abstract

This study investigates the joint replenishment decisions for a central factory and its satellite factories on a Just in Time (JIT) basis. A joint replenishment program involves combining different materials/components from several satellite factories to form a large shipment delivered to the central factory, which provides an opportunity to take advantage of the strong economies of scale present in the freight cost. The program requires closely coordinating the inventory control across satellite factories of multiple materials/components. A decision model with a solution procedure is provided to find the optimal common replenishment cycle time of a group of materials/components for the central factory and the optimal production cycle time as well as the number of shipments per production cycle for each satellite factory. Based on the presented solution procedure, a decision support system has been implemented on a personal computer for the proposed decision. Numerical experiments were conducted to illustrate the application of the model.

Keywords: Inventory control; Joint replenishment; Central factory; Satellite factory

1. Introduction

With JIT purchasing, suppliers become extended operations of the JIT manufacturer. The importance of JIT purchasing in the overall JIT management system is demonstrated by the magnitude of parts by a typical manufacturer. In 1985, purchased materials for all US manufacturers amounted to 60% of total sales revenue. Japanese manufacturers typically have a much higher percentage of total manufacturers (White & Pearson, 1994). Researches and applications concerning JIT purchasing techniques have been reported in various studies (Giunipero, 1986; Hahn, Pinto, & Bragg, 1993; Schonberger & Ansari, 1984). The success and resulting performance of purchasing system is based upon cooperation between the purchaser and supplier. JIT purchasing practices are usually characterized by a central factory with a number of satellite factories which are located close and supply materials/components to the central factory, make frequent deliveries, and are considered long-term partners with the central factory. Under these operating conditions, the relations between the central and the satellite factories are built on a high degree of mutual trust and openness. Both the central and the satellite factories must share information and protect its confidentiality.

Freight cost makes a significant portion of the cost incurred for most purchasing. Consolidating the replenishments of multiple materials/components has been a trend in JIT purchasing since it offers the possibility of further reducing freight costs by combining purchases across materials/components. Each time an order is issued, a major ordering cost, independent of the number of materials/components ordered, for transportation and order processing will be incurred. Joint replenishment of a group of materials/components reduces the occurrences that a major ordering cost is charged, and consequently saves costs. There are already many case reports on firms streamlining their procurements and inventory management by consolidating and centralizing the replenishment of several

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materials/components (Lawrence & Varma, 1999). However, joint replenishment is not an easy program to implement. It requires close relationships between the buyer and multiple suppliers. Therefore, to operate efficiently in the joint replenishment environment, it is necessary to optimize the related activities of both the central factory and its satellite factories simultaneously.

Regarding the research of inventory control with joint replenishment, Swenseth and Park (1993) pioneered an inventory model for the case of a single-buyer and multiple-supplier. However, the assumption, which the delivery initiates only after the production run finished, restricts the application of the model. Later, Kohli and Park (1994) examined efficient transactions across multiple products with suppliers running under a lot-for-lot replenishment policy. Qu, Bookbinder, and Iyogun (1999) dealt with an inbound material-collection problem for an integrated inventory-transportation system with a modified periodic-review inventory policy and a traveling-salesman component.

Another line of the literature related to the issue is the research of joint replenishment problem (JRP). Over the last two decades, the JRP has received much attention of researchers. The JRP is usually based on a buyer-only viewpoint with concerning multiple products where economies exist for replenishing products collectively. The problem involves determining a basic replenishment cycle time T and the replenishment interval k_iT for item i, where k_i is an integral number. The objective function of the JRP is not convex and typically has several local minima. Optimal algorithms enumerate all the local minimum solutions between a lower bound and a upper bound for T. A decision model with a solution approach for the JRP was first presented by Goyal (1974). This approach is based on enumeration and the running time of the procedure grows exponentially with the number of replenished items. Van Eijs (1993) proposed a modified lower bound that guarantees an optimal solution for a joint replenishment problem. However, the resulting algorithm often requires even more computation time than the algorithm of Goyal (1974), except for large values of the major ordering cost. Viswanathan (1996) presented a new algorithm that reduces the computational effort required by that in Van Eijs (1993) with improving the bounds on the basic cycle time T and iterative fashion. Fung and Ma (2001) proposed a new approach for determining the optimal policy for a JRP. Viswanathan (2002) suggested a numerical study to compare the computational efficiency of the algorithm of Fung and Ma (2001) with the algorithm proposed by Viswanathan (1996). Recently, Siajadi, Ibrahim, Lochert, and Chan (2005) investigated an integrated inventory system where a single manufacturer consumes and purchases raw materials from a single supplier in order to produce a specific finished item.

The study is to model the decision that determines replenishment policies for a central factory and its satellite factories which collaborate in implementing a joint replenishment program under the JIT concept. The delivery restriction made by Swenseth and Park (1993) is released. A common shipment cycle is needed to collect multiple materials/components from certain satellite factories. The lot size of each material production run is considered that it can be delivered in several number shipments to the central factory. The problem for the central-satellite factories is to determine the common pick-up cycle time and the multiple for settling the number of shipments per production cycle for each material to optimize their joint total cost. A modification of the JRP algorithm of Fung and Ma (2001) is presented to determine the optimal solution to this replenishment decision. Based on the presented solution procedure, a decision support system has been implemented on a personal computer for the proposed decision.

The paper proceeds as follows. In the next section, the notation and underlying assumptions are introduced. In Section 3, the model to analyze the coordinated replenishment decisions as well as the algorithm for solving the model is presented. Based on the solution algorithm described in Section 3, the decision support system along with numerical cases is introduced in Section 4 to illustrate the application of the proposed model. Finally, the concluding remarks are made in the last section.

2. Notation and assumptions

To develop the decision model, the following notation is adopted and the assumptions made in this study are introduced.

Notation

n	number of materials involved in the joint replenishment program
D_i	demand rate of material i
P_i	production rate of material i in a satellite factory, $P_i \geqslant D_i$
T	the common shipment cycle time for the n materials
T_i	the shipment cycle time of the material i with individual replenishment
k_i	the number of shipments in which the material i is delivered from the satellite factory to the central factory within one production cycle, a positive integer
S_i	setup cost of material i for the satellite factory
F	major ordering cost for the central factory
F_i	minor ordering cost of material i for the central factory
H_{si}	carrying cost rate of material i for the satellite factory
H_{ci}	carrying cost rate of material i for the central factory
TC_{si}	total relevant cost of material i per year for the satellite factory
TC_{ci}	total relevant cost of material i per year for the central factory

JTC_0 total relevant cost for the central and the satellite factories per year with individual replenishment of materials

JTC joint total relevant cost for the central and the satellite factories per year with joint replenishment of materials

Assumptions:

This study considers that a central factory replenishes n materials from a number of satellite factories which are located close to and are considered long-term partners with the central factory. The central and the satellite factories share information of production and inventory status. The behavior of the inventory levels of the central and the satellite factories with joint replenishment of materials is illustrated in Fig. 1. The demand occurs at a constant rate of D_i per year for material i ($i = 1,2,\ldots,n$) with no shortage allowed. Each time the central factory has a need for replenishment, a major ordering cost F, regardless of the number of the materials included, and a minor ordering cost F_i, related to material i, are incurred. The carrying costs are incurred to the central and the satellite factories at rates of H_{ci} and H_{si}, respectively, per year per unit of material i held. The production rate for material i is P_i with a setup cost S_i for each production run. The quantity in one product cycle of material i is split into k_i equal-size shipment lots delivered to the central factory. When the replenishment of the n materials is consolidated, the independently determined shipment cycle time T_i for material i will be replaced with a joint determined shipment cycle time T.

3. Model formulation and solution procedures

First, the case of individual replenishment of the n materials is considered. For any given k_i, according to the study of Goyal and Srinivasan (1992), the optimal independent shipment cycle time T_i for material i can be determined by

$$T_i^*(k_i) = \left\{ \frac{2\left(F + F_i + \frac{S_i}{k_i}\right)}{D_i\left[H_{ci} + H_{si}\left(\frac{2D_i}{P_i} - 1\right) + H_{si}\left(1 - \frac{D_i}{P_i}\right)k_i\right]} \right\}^{\frac{1}{2}}, \quad (1)$$

and the optimal solution of k_i can be derived by

$$k_i^*\left(k_i^* - 1\right) < S_i\left[H_{ci} + H_{si}\left(\frac{2D_i}{P_i} - 1\right)\right] \Big/ \left[H_{si}(F + F_i)\left(1 - \frac{D_i}{P_i}\right)\right] \leqslant k_i^*\left(k_i^* + 1\right). \quad (2)$$

Then the minimal total cost per year incurred to the central and the satellite factories for independently replenishing the n materials can be derived as

$$JTC_0^* = \sum_{i=1}^{n} \sqrt{2D_i\left(F + F_i + \frac{S_i}{k_i^*}\right)\left[H_{ci} + H_{si}\left(\frac{2D_i}{P_i} - 1\right) + H_{si}\left(1 - \frac{D_i}{P_i}\right)k_i^*\right]}. \quad (3)$$

In the case of joint replenishment, the total cost per year for the central factory is

$$TC_c = \frac{1}{T}\sum_{i=1}^{n}(F + F_i) + \frac{T}{2}\sum_{i=1}^{n} D_i H_{ci},$$

and the total cost of material i per year for the satellite factory is

$$TC_{si} = \frac{1}{T}\sum_{i=1}^{n}\frac{S_i}{k_i} + \frac{T}{2}\sum_{i=1}^{n} D_i H_{si}\left[\frac{2D_i}{P_i} - 1 + \left(1 - \frac{D_i}{P_i}\right)k_i\right].$$

Then, the joint total cost per year for the central factory and all satellite factories involved in the joint replenishment program is

$$\begin{aligned} JTC(T,K) &= TC_c + \sum_{i=1}^{n} TC_{si} \\ &= \frac{1}{T}\left(A + \sum_{i=1}^{n}\frac{S_i}{k_i}\right) \\ &\quad + \frac{T}{2}\left[B + \sum_{i=1}^{n} D_i H_{si}\left(1 - \frac{D_i}{P_i}\right)k_i\right], \end{aligned} \quad (4)$$

where $K = (k_1,k_2,\ldots,k_n)$, $A = F + \sum_{i=1}^{n} F_i$, and $B = \sum_{i=1}^{n} D_i\left[H_{ci} + H_{si}\left(\frac{2D_i}{P_i} - 1\right)\right]$.

Now, the problem is to find solutions for $T, k_1, k_2, \ldots,$ and k_n in (4) so that JTC can be minimized. For any fixed $K = (k_1,k_2,\ldots,k_n)$, it can be shown that the corresponding optimal common shipment cycle time $T^*(K)$ is given by

$$T^*(K) = \sqrt{\frac{2\left(A + \sum_{i=1}^{n}\frac{S_i}{k_i}\right)}{B + \sum_{i=1}^{n} D_i H_{si}\left(1 - \frac{D_i}{P_i}\right)k_i}}. \quad (5)$$

By substituting (5) into (4), there is

$$JTC(K) = \sqrt{2\left(A + \sum_{i=1}^{n}\frac{S_i}{k_i}\right)\left[B + \sum_{i=1}^{n} D_i H_{si}\left(1 - \frac{D_i}{P_i}\right)k_i\right]}. \quad (6)$$

For any fixed value of T, the optimal solution of k_i must satisfy

$$k_i(T)(k_i(T) - 1) < \frac{2S_i/D_i H_{si}\left(1 - \frac{D_i}{P_i}\right)}{T^2} \leqslant k_i(T)(k_i(T) + 1). \quad (7)$$

This can be rearranged as

$$\frac{2S_i/D_i H_{si}\left(1 - \frac{D_i}{P_i}\right)}{k_i(T)(k_i(T) + 1)} \leqslant T^2 < \frac{2S_i/D_i H_{si}\left(1 - \frac{D_i}{P_i}\right)}{k_i(T)(k_i(T) + 1)}.$$

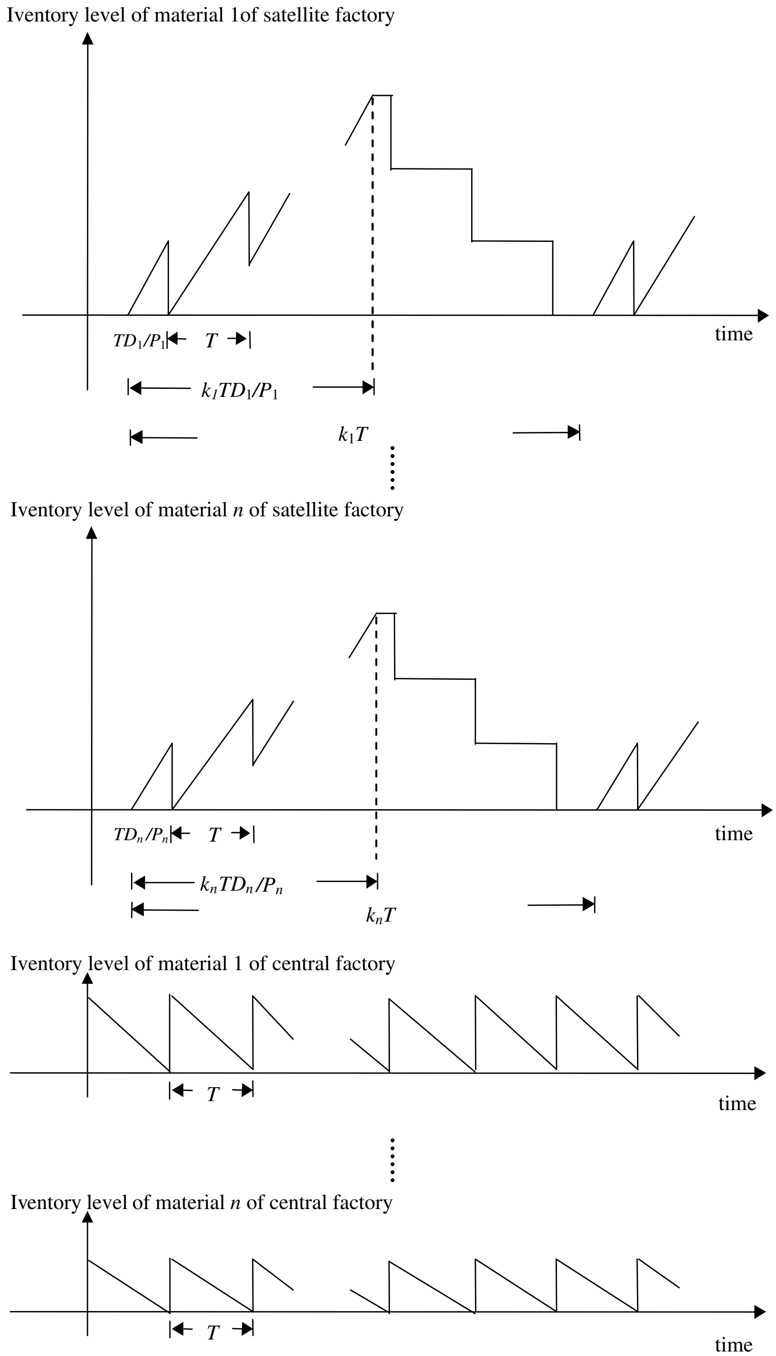

Fig. 1. Inventory levels of the central factory and the satellite factories with joint replenishment.

Van Eijs (1993) suggested a lower bound for T^* as $2A/\text{JTC}^*$. Fung and Ma (2001) proposed a pair of bounds for T^* as

$$T_{\min} = 2\left(A + \sum_{i=1}^{n} \frac{S_i}{k_{i\max}}\right)/\text{JTC}(K). \tag{8}$$

and

$$T_{\max} = \frac{\text{JTC}(K)}{B + \sum_{i=1}^{n} D_i H_{si}\left(1 - \frac{D_i}{P_i}\right) k_{i\min}}. \tag{9}$$

Now, suppose that there are n materials involved in the joint replenishment program. Since the Steps 8 and 9 in the algorithm of Fung and Ma (2001) could not guarantee the derived solution to be optimal for (6), the algorithm of Fung and Ma (2001) is modified and stated as follows. In the algorithm, an initial value of K^* is determined in Step 3 which is based on the heuristic algorithm of Silver, Pyke, and Peterson (1998) and a new rule for determining the optimum for the problem (see Appendix for proof) is presented in Step 11 and Step 12.

Algorithm:

Step 1: Input parameters D_i, P_i, S_i, F, F_i, H_{si}, H_{ci} $(i = 1,2,\ldots,n)$.

Step 2: Calculate $A = F + \sum_{i=1}^{n} F_i$, and $B = \sum_{i=1}^{n} D_i \left[H_{ci} + H_{si}\left(\frac{2D_i}{P_i} - 1\right)\right]$.

Step 3: Obtain an initial value of $K^* = (k_1, k_2, \ldots, k_n)$.

Step 3.1: Number the items such that $\frac{S_i}{D_i H_{si}\left(1-\frac{D_i}{P_i}\right)}$ is the smallest for material 1. Set $k_1 = 1$.

Step 3.2: Evaluate $k_j = \sqrt{\frac{S_j D_1 H_{s1}\left(1-\frac{D_1}{P_1}\right)}{D_j H_{sj}\left(1-\frac{D_j}{P_j}\right)(A+S_1)}}$ rounded to the nearest integer greater than zero, $j = 2,\ldots,n$.

Step 4: Substituting K^* into (6) to get JTC*, and let $T_{\max} = \frac{\mathrm{JTC}^*}{B+\sum_{i=1}^{n} D_i H_{si}\left(1-\frac{D_i}{P_i}\right)}$.

Step 5: Substitute $T_{\max}$ into (7) to get $K_{\min}$. Substitute $K_{\min}$ into (6) to obtain $JTC_{\min}$.

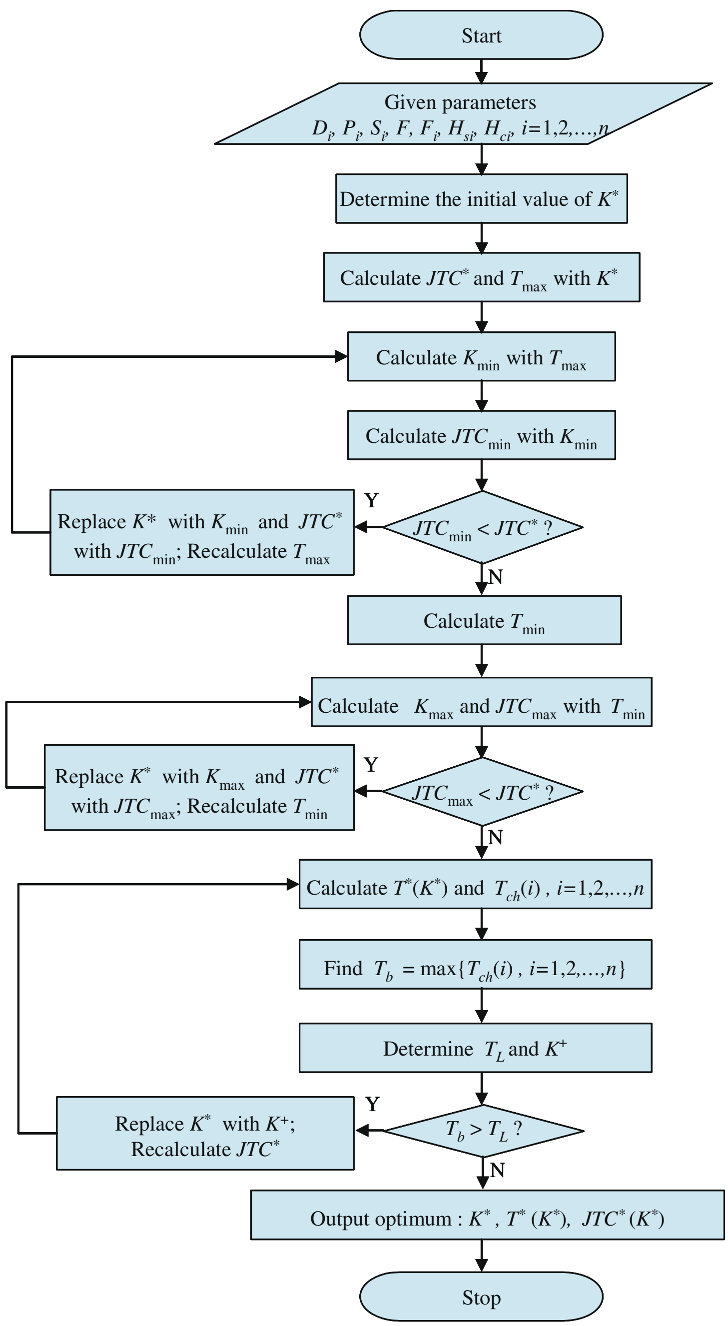

Fig. 2. Flow chart of the solution procedure to find the optimal joint replenishment policy.

Step 6: If $\text{JTC}_{\min} < \text{JTC}^*$, let $K^* = K_{\min}$, $\text{JTC}^* = \text{JTC}_{\min}$, and $T_{\max} = \frac{\text{JTC}_{\min}}{B+\sum_{i=1}^{n} D_i H_{si}\left(1-\frac{D_i}{P_i}\right) k_{i\min}}$, go to Step 5; Else go to Step 7.
Step 7: Let $T_{\min} = 2A/\text{JTC}^*$.
Step 8: Substitute $T_{\min}$ into (7) and (6) to obtain $K_{\max}$ and $\text{JTC}_{\max}$, respectively.
Step 9: If $\text{JTC}_{\max} < \text{JTC}^*$, let $K^* = K_{\max}$, $\text{JTC}^* = \text{JTC}_{\max}$, and $T_{\min} = \frac{2\left(A+\sum_{i=1}^{n} S_i/k_i^{\max}\right)}{\text{JTC}_{\max}}$, go to Step 8; Else substitute K^* into (5) to obtain $T^*(K^*)$ and go to Step 10.
Step 10: Let $T_{\text{ch}}(i) = \sqrt{2S_i/D_i H_{si}\left(1-\frac{D_i}{P_i}\right) k_i^*(k_i^*+1)}$, $i = 1, 2, \ldots, n$.
Step 11: Let $T_b = \max_{1 \leqslant i \leqslant n} T_{\text{ch}}(i)$.
Step 12: Let $P = \{p \mid T_{\text{ch}}(p) = T_b,\ 1 \leqslant p \leqslant n\}$ and

$$T_L = \sqrt{\frac{\sum_{i=1}^{n}\left[B+D_i H_{si}\left(1-\frac{D_i}{P_i}\right)\right]k_i^*}{\sum_{i=1}^{n}\left[B+D_i H_{si}\left(1-\frac{D_i}{P_i}\right)\right]k_i^+}}\,T^*(K^*),$$

where $k_i^+ = k_i^*$, $i = 1, 2, \ldots, n$ and $i \neq p$, $k_p^+ = k_p^* + 1$, $p \in P$.
Step 13: If $T_b > T_L$, let $K^* = K^+$, $\text{JTC}^* = \text{JTC}(K^+)$, and $T^* = T^*(K^+) = \frac{2\left(A+\sum_{i=1}^{n} S_i/k_i^+\right)}{\text{JTC}(K^+)}$, go to Step 10; Else go to Step 14.
Step 14: Stop the algorithm by letting the optimal solution for K be K^* with the corresponding T^* and JTC^*.

The flow chart of the solution procedure to find the optimal joint replenishment decision has been depicted in Fig. 2.

4. Computer implementation

The algorithm presented in Section 3 has been implemented as a decision support system on a personal computer at which Intel Pentium(R) IV CPU 2.80 GHz, 768MB DDR RAM inside it. Visual Basic 2005 is utilized as the software platform to develop the decision support system. Fig. 3 shows the window to input the number of materials, e.g., $n = 3$ in this case, involved in the joint

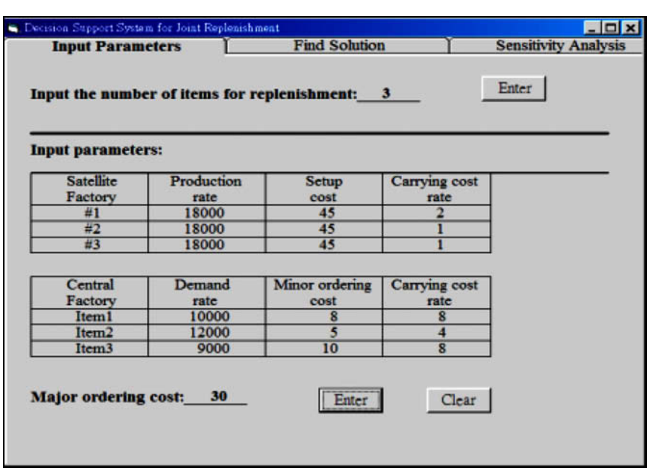

Fig. 3. The system window to input parameters.

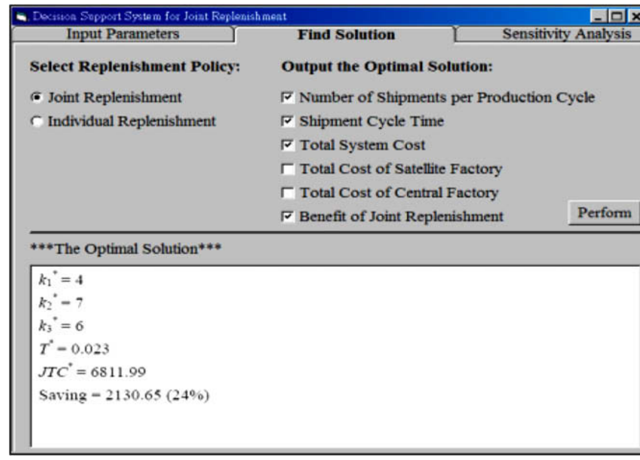

Fig. 4. The optimal solution for the joint replenishment policy provided by the decision support system.

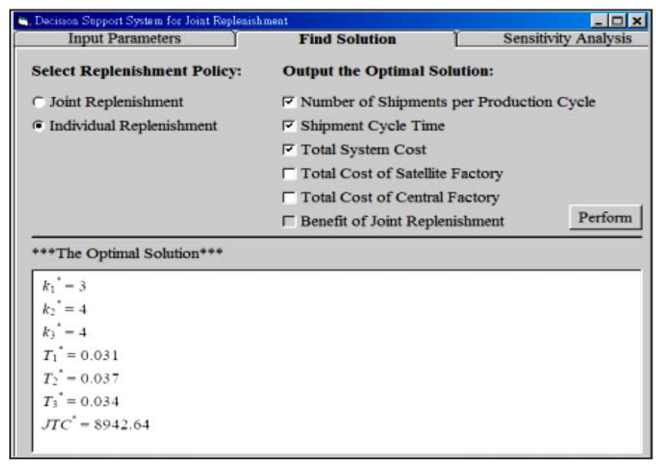

Fig. 5. The optimal solution for the individual replenishment policy provided by the decision support system.

replenishment decision. Once the decision maker inputs the number of materials, a window to specify the relevant parameters will be displayed as shown in Fig. 3. If all of the relevant parameters are input for the system, the decision maker then clicks the label "Find Solution" to select the replenishment policy and to set the options of output. Fig. 4 illustrates the result that the system derives the optimal solution in which $K^* = (4, 7, 6)$, $T^* = 0.023$ (year), and $\text{JTC}^* = \$6811.99$ for the joint replenishment policy after the decision maker clicks "Perform". From the value of "Saving", the benefit by adopting the joint replenishment policy instead of the individual replenishment policy can be learned. As shown in Fig. 5, the system also provides the optimal inventory decision for the case of individual replenishment of the n materials. One advantage of the system is that the decision maker can learn the sensitivity of replenishment decision and the joint total cost by specifying the parameter with the variation range in the Window of "Sensitivity Analysis". Fig. 6 illustrates the output of the sensitivity analysis of K^*, T^*, JTC^*, and saving for the variation of major ordering cost.

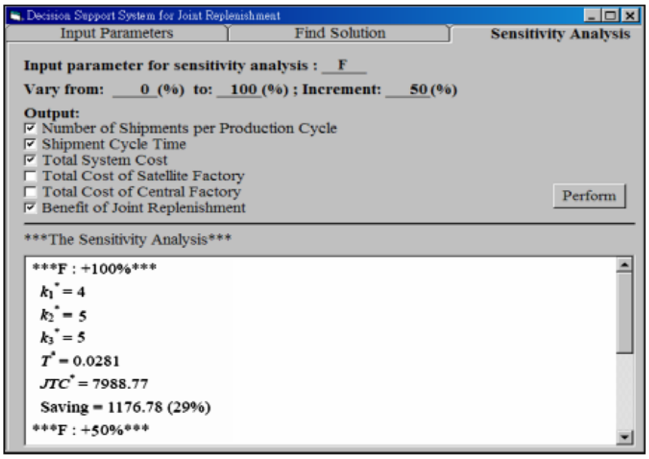

Fig. 6. The sensitivity analysis function of the decision support system.

The sensitivity of the optimal solution has been further examined by conducting several numerical experiments. The results are illustrated by a base case and are shown in Table 1. Some findings are summarized as follows:

1. The joint total cost of the central and the satellite factories can be significantly improved (approximately by 24% in the base case), the shipment cycle times for all materials will be shortened, and the deliveries from the satellite factories to the central factory may become more frequent within a production cycle after the replenishment consolidation program is implemented.
2. When the production rate P or the minor ordering cost F_i increases, the production cycle times of satellite factories and the improvement rate of the joint total cost ΔJTC will not increase. A larger P may result in a larger carrying cost for the satellite factory, which will encourage a smaller production lot size and thus a shorter production cycle time. A larger F_i implies a larger fraction of the minor ordering cost to the overall ordering costs, which makes the return of replenishment consolidation to be less significant.
3. When the major ordering F increases, the value of K^* will not increase and the values of T^* and ΔJTC will increase. The result implies that the larger the major ordering cost, the more beneficial to the integrated central-satellite factories from implementing the replenishment consolidation program.
4. The increase of the value of H_{ci} will result in a smaller common shipment cycle time and a higher improvement rate of the joint total cost. In contrast, as the value of H_{si} increases, the value of ΔJTC decreases.
5. In general, the inventory replenishment decisions K^* and T^* are especially sensitive to the variation of production rate and carrying cost rates of the system. Among all of the parameters, the value of ΔJTC is most sensitive to the variation of major and minor ordering costs and the carrying cost rate of central factory.

5. Concluding and remarks

In this study, the decision of joint replenishment of several materials for a production system consisting of a central and multiple satellite factories is modeled. Based on the concept JIT purchasing, the model considers the situation where a central factory coordinates its multiple satellite factories to replenish certain materials at a common shipment cycle time to minimize their joint total cost. The problem is to determine the optimal common shipment cycle time and the number of shipments per production cycle for each material involved in the replenishment con-

Table 1
An illustration of the sensitivity analysis for parameters

	K^*	T^*	JTC*	K_0^{*a}	T_0^{*b}	JTC_0^*	ΔJTC(%)[c]
Base case	(4,7,6)	0.0230	6811.99	(3, 4, 4)	(0.031, 0.037, 0.034)	8942.64	24
P_1							
$P_2 + 100\%$	(3,4,5)	0.0246	7175.24	(2,3,3)	(0.035,0.038,0.035)	9193.03	22
P_3							
S_1							
$S_2 + 100\%$	(6,9,9)	0.0229	7690.63	(5,6,6)	(0.030,0.036,0.033)	9823.33	22
S_3							
$F + 100\%$	(4,5,5)	0.0281	7988.77	(3,3,3)	(0.039,0.050,0.045)	11264.02	29
F_1							
$F_2 + 100\%$	(4,6,5)	0.0268	7734.19	(3,4,4)	(0.033,0.039,0.037)	9605.25	19
F_3							
H_{s1}							
$H_{s2} + 100\%$	(3,5,4)	0.0227	7766.84	(2,3,3)	(0.032,0.035,0.033)	9938.94	22
H_{s3}							
H_{c1}							
$H_{c2} + 100\%$	(6,9,9)	0.0162	8693.90	(5,6,6)	(0.021,0.026,0.024)	11684.60	26
H_{c3}							

Base case: $D_1 = 10{,}000$, $D_2 = 12{,}000$, $D_3 = 9000$, $P_1 = P_2 = P_3 = 18{,}000$, $S_1 = S_2 = S_3 = 45$, $F = 30$, $F_1 = 8$, $F_2 = 5$, $F_3 = 10$, $H_{s1} = 2$, $H_{s2} = 1$, $H_{s3} = 1$, $H_{c1} = 8$, $H_{c2} = 4$, $H_{c3} = 8$.

[a] K_0^* is the set of optimal solution of k_i $(i = 1,2,3)$ for the individual replenishment policy.

[b] T_0^* is the set of optimal solution of T_i $(i = 1,2,3)$ for the individual replenishment policy.

[c] $\Delta JTC = \left[\left(JTC_0^* - JTC^*\right)/JTC_0^*\right] \times 100$, where JTC_0^* is the minimal joint total cost for the individual replenishment policy.

solidation program. A modified algorithm of Fung and Ma (2001) is proposed and applied to ensure finding the optimal solution for the problem.

Based on the algorithm, a decision support system has been implemented and several numerical experiments were conducted to illustrate the application of the proposed model. It is apparent, compared to independent replenishment, that a significant reduction in joint total cost can be achieved by integrating replenishments of multiple materials. In addition, numerical results show that the integrated central-satellite factories may gain a substantial saving in joint total cost from replenishment consolidation, and that the saving increases as the major ordering cost or the carrying cost rate of the central factory increases or as the production rate, the setup cost, or the carrying cost rate of the satellite factory decreases.

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Appendix

To ensure finding the optimal solution, the following Lemma must be established.

Lemma 1. *If and only if* $T_b > T_L$, *there is* $JTC(K^+) < JTC(K^*)$.

Proof. Let $T_{\text{ch}}(i) = \sqrt{2S_i/D_iH_{\text{si}}\left(1-\frac{D_i}{P_i}\right)k_i^*(k_i^{*+1})}$, $T_b = \max_{1\leqslant i\leqslant n} T_{\text{ch}}(i)$,

$P = \{p|T_{\text{ch}}(p) = T_b,\ p = 1,2,\ldots,n\}$, and T_L

$$= \sqrt{\frac{\sum_{i=1}^{n}\left[B + D_iH_{si}\left(1-\frac{D_i}{P_i}\right)\right]k_i^*}{\sum_{i=1}^{n}\left[B + D_iH_{si}\left(1-\frac{D_i}{P_i}\right)\right]k_i^+}T^*(K^*)},$$

where $k_i^+ = k_i^*, i = 1,2,\ldots,n$ and $i \neq p, k_p^+ = k_p^* + 1,\ p \in P$.

Substituting K^+ into (3) and taking square of the result, we can yield

$$\begin{aligned}
\text{JTC}^2(K^+) &= 2\left(A+\sum_{i=1}^{n}\frac{S_i}{k_i^+}\right)\left[B+\sum_{i=1}^{n}D_iH_{\text{si}}\left(1-\frac{D_i}{P_i}\right)k_i^+\right]\\
&= 2\left(A+\sum_{i=1\ i\neq p}^{n}\frac{S_i}{k_i^*}+\frac{S_p}{k_{p+1}}\right)\left[B+\sum_{i=1\ i\neq p}^{n}D_iH_{\text{si}}\right.\\
&\quad\left.\times\left(1-\frac{D_i}{P_i}\right)k_i^*+D_pH_{sp}\left(1-\frac{D_p}{P_p}\right)\left(k_p^*+1\right)\right]\\
&= 2\left[A+\sum_{i=1}^{n}\frac{S_i}{k_i^*}-S_p\left(\frac{1}{k_p^*}-\frac{1}{k_{p+1}^*}\right)\right]\\
&\quad\times\left[B+\sum_{i=1}^{n}D_iH_{\text{si}}\left(1-\frac{D_i}{P_i}\right)k_i^*+D_pH_{sp}\left(1-\frac{D_p}{P_p}\right)\right]\\
&= 2\left(A+\sum_{i=1}^{n}\frac{S_i}{k_i^*}\right)\left[B+\sum_{i=1}^{n}D_iH_{\text{si}}\left(1-\frac{D_i}{P_i}\right)k_i^*\right]\\
&\quad+2\left(A+\sum_{i=1}^{n}\frac{S_i}{k_i^*}\right)\left[D_pH_{sp}\left(1-\frac{D_p}{P_p}\right)\right]\\
&\quad-\frac{2S_p}{k_p^*\left(k_p^*+1\right)}\left[B+\sum_{i=1}^{n}D_iH_{\text{si}}\left(1-\frac{D_i}{P_i}\right)k_i^+\right]\\
&= \text{JTC}^2(K^*)+2\left(A+\sum_{i=1}^{n}\frac{S_i}{k_i^*}\right)\left[D_pH_{sp}\left(1-\frac{D_p}{P_p}\right)\right]\\
&\quad-\frac{2S_p}{k_p^*\left(k_p^*+1\right)}\left[B+\sum_{i=1}^{n}D_iH_{\text{si}}\left(1-\frac{D_i}{P_i}\right)k_i^+\right].
\end{aligned}$$

Then,

$$\begin{aligned}
&\text{JTC}^2(K^+)-\text{JTC}^2(K^*)\\
&= 2\left(A+\sum_{i=1}^{n}\frac{S_i}{k_i^*}\right)\left[D_pH_{sp}\left(1-\frac{D_p}{P_p}\right)\right]\\
&\quad-\frac{2S_p}{k_p^*\left(k_p^*+1\right)}\left[B+\sum_{i=1}^{n}D_iH_{\text{si}}\left(1-\frac{D_i}{P_i}\right)k_i^+\right]\\
&= D_pH_{sp}\left(1-\frac{D_p}{P_p}\right)\left[B+\sum_{i=1}^{n}D_iH_{\text{si}}\left(1-\frac{D_i}{P_i}\right)k_i^+\right]\\
&\quad*\left[\frac{2\left(A+\sum_{i=1}^{n}\frac{S_i}{k_i^*}\right)}{B+\sum_{i=1}^{n}D_iH_{\text{si}}\left(1-\frac{D_i}{P_i}\right)k_i^+}-\frac{2S_p}{D_pH_{sp}\left(1-\frac{D_p}{P_p}\right)k_p^*\left(k_p^*+1\right)}\right]\\
&= D_pH_{sp}\left(1-\frac{D_p}{P_p}\right)\left[B+\sum_{i=1}^{n}D_iH_{\text{si}}\left(1-\frac{D_i}{P_i}\right)k_i^+\right]\\
&\quad\times\left[\frac{B+\sum_{i=1}^{n}D_iH_{\text{si}}\left(1-\frac{D_i}{P_i}\right)k_i^*}{B+\sum_{i=1}^{n}D_iH_{\text{si}}\left(1-\frac{D_i}{P_i}\right)k_i^+}(T^*(K^*))^2-T_b^2\right]\\
&= D_pH_{sp}\left(1-\frac{D_p}{P_p}\right)\left[B+\sum_{i=1}^{n}D_iH_{\text{si}}\left(1-\frac{D_i}{P_i}\right)k_i^+\right]\left(T_L^2-T_b^2\right).
\end{aligned}$$

Consequently, $\text{JTC}(K^+) < \text{JTC}(K^*)$ if and only if $T_b > T_L$. □

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