# Complements on Monte Carlo methods

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**Exercise 1** We want to estimate

$$I = \int_0^1 (1+x)^{-1/2} \exp(-x) \mathrm{d}x \,.$$

**1** Propose two estimators of $I$ using different Monte Carlo methods:

- the first strategy will be based on a uniform distribution and the corresponding estimator will be denoted $\hat{I}_N^1$ where $N$ is the number of simulations;
- the second method will be based on an exponential distribution and the associated estimator will be denoted $\hat{I}_N^2$ with $N$ the number of simulations.

**2** Compare $\hat{I}_N^1$ and $\hat{I}_N^2$.

**3** Give an asymptotic confidence interval of level 95% for the parameter $I$. Hint: use the central limit theorem associated with $I_N^1$.

**4** Using the Bienaymé-Chebyshev inequality, determine the number of simulations simulation necessary for the relative absolute error of $\hat{I}_N^1$ to be less than 1% with a probability greater than or equal to 99%.

**Exercise 2** We want to estimate

$$I = \int_2^{+\infty} \exp\left(x - x^2/2\right) \mathrm{d}x \,.$$

**1** Propose an estimator $\hat{I}_1^N$ of $I$ based on the simulation of $N$ random variables from a $\mathcal{N}(0,1)$.

**2** Propose an estimator $\hat{I}_2^N$ of $I$ based on the simulation of $N$ random variables from a $\mathcal{N}(1,1)$.

**3** Compare the variances of $\hat{I}_1^N$ and $\hat{I}_2^N$.

**Exercise 3** We want to estimate

$$I_l = \mathbb{P}_{Exp(1)}\left(X \in [l, l+1]\right) .$$

**1** Propose an estimator $\hat{I}_l^1$ of $I_l$ based on a Monte-Carlo method using the exponential distribution.

**2** Propose an estimator $\hat{I}_l^2$ of $I_l$ based on a Monte-Carlo method using the uniform distribution.

**3** Compare $\hat{I}_l^1$ and $\hat{I}_l^2$ according to the values of $l$.

**Exercise 4** Let $g$ be a bounded integrable function ($0 \le g \le 1$) on $[0, 1]$.
We want to estimate

$$m = \int_0^1 g(x)dx .$$

Let $X$ and $Y$ be two independent uniformly distributed variables on $[0, 1]$.
Let

$$U = \mathbf{1}_{Y \le g(X)}, \qquad V = g(X) \quad \text{and} \quad W = \frac{g(X) + g(1-X)}{2} .$$

**1** Calculate the mean and variance of $U$, $V$ and $W$. Compare the variances.

**2** Suggest three Monte-Carlo methods for estimating $m$.

**3** In what follows, assume that $g$ is monotone. Check that $\mathbb{E}(g(X)g(1-X)) \le m^2$.

**4** Consider $(X_i)_{i\ge1}$ a sequence iid of variables with a uniform distribution on $[0, 1]$.
Let $A_n = \dfrac{1}{2n}\displaystyle\sum_{i=1}^{2n} g(X_i)$ and $B_n = \dfrac{1}{2n}\displaystyle\sum_{i=1}^{n}(g(X_i) + g(1-X_i))$, which is the better estimator of $m$?