Monte Carlo and Markov chain Monte Carlo methods

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Standard Monte Carlo method

General definition use of randomness to solve a problem centered on a calculation

There is no consensus to give a more precise definition

Methods that have been used for centuries: traces as far away as in Babylon and the Old Testament!

Standard Monte Carlo method

[1733, Buffon's Needle] give an approximate value to π

Throw a l long needle on a floor of parallel slats that create d widths with $l \leqslant d$

If the needle is thrown uniformly on the ground (to be specified!), the probability that it intersects with one of the joins between the slats is $\dfrac{2l}{\pi d}$

If you make several independent rolls and you note p the proportion of tests that hit one of the straight lines forming the separations between the slats, π can be estimated by $\dfrac{2l}{pd}$

Standard Monte Carlo method

[World War II, Los Alamos: Ulam, Metropolis and von Neumann] preparation of the first atomic bomb

The Monte Carlo appellation is due to Metropolis, inspired by Ulam's interest in poker

Work at Los Alamos: directly simulate neutron dispersion and absorption problems for fissile materials

Standard Monte Carlo method

Theorem (strong law of large numbers) Let $(X_n)_{n\in\mathbb{N}}$ be an iid sequence of random variables with probability distribution f
If $\mathbb{E}_f(|X_i|) < \infty$

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow_{\text{ps}} \mathbb{E}_f(X_1)$$

Standard Monte Carlo method

Theorem (central limit theorem) Let $(X_n)_{n\in\mathbb{N}}$ be an iid sequence of random variables with probability distribution f
If $\mathbb{E}_f(|X_i|^2) < \infty$

$$\sqrt{n}\left(\frac{\bar{X}_n - \mathbb{E}_f(X_1)}{\sqrt{\mathbb{V}_f(X_1)}}\right) \longrightarrow_{\mathscr{L}} N(0,1)$$

Standard Monte Carlo method

Target

$$\mathbb{E}_f(h(X)) = \int h(x)f(x)d\mu(x) < \infty$$

(f is the density of X with respect to μ)

Standard Monte Carlo estimator of $\mathbb{E}_f(h(X))$

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i)$$

where $X_1, \ldots, X_n$ is an iid sample from f

Standard Monte Carlo method

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i) \longrightarrow_{\text{ps}} \mathbb{E}_f(h(X))$$

$$\mathbb{E}_{f^{\otimes n}}\left(\frac{1}{n}\sum_{i=1}^{n} h(X_i)\right) = \mathbb{E}_f(h(X))$$

Standard Monte Carlo method

$$\mathbb{V}_{f^{\otimes n}}\left[\frac{1}{n}\sum_{i=1}^{n} h(X_i)\right] = \frac{1}{n}\mathbb{V}_f(h(X))$$

$$\frac{1}{n}\left[\frac{1}{n-1}\sum_{i=1}^{n}\left(h(X_i) - \frac{1}{n}\sum_{j=1}^{n} h(X_j)\right)^2\right]$$

is an unbiased estimator of $\mathbb{V}_f(h(X))/n$

Standard Monte Carlo method

If $\mathbb{E}_f(|h(X)|^2) < \infty$

$$\frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} h(X_i) - \mathbb{E}_f(h(X))\right)}{\sqrt{\mathbb{V}_f(h(X))}} \longrightarrow_{\mathscr{L}} N(0,1)$$

Standard Monte Carlo method

Convergence speed for various quadrature rules and for the Monte Carlo method in s dimension and using n points

- Trapezoidal rule: $n^{-2/s}$
- Simpson rule: $n^{-4/s}$
- Gauss rule with m points: $n^{-(2m-1)/s}$
- Monte-Carlo method: $n^{-1/2}$

Importance Sampling methods

Target

$$\mathbb{E}_f(h(X)) = \int h(x)f(x)d\mu(x) < \infty$$

We consider the probability density g (with respect to μ) such that: if $g(x) = 0$ then $f(x)|h(x)| = 0$

Importance Sampling methods

$$\mathbb{E}_f(h(X)) = \int h(x) f(x) d\mu(x) =$$

$$\int h(x) \frac{f(x)}{g(x)} g(x) d\mu(x) = \mathbb{E}_g \left[h(X) \frac{f(X)}{g(X)} \right]$$

Importance sampling estimator of $\mathbb{E}_f(h(X))$

$$\frac{1}{n} \sum_{i=1}^{n} h(X_i) \frac{f(X_i)}{g(X_i)}$$

where $X_1, \ldots, X_n$ **is an iid sample from** g

Importance Sampling methods

If $f|h|$ is absolutely continuous with respect to g

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i)\frac{f(X_i)}{g(X_i)} \longrightarrow_{\text{ps}} \mathbb{E}_f(h(X))$$

is convergent

$$\mathbb{E}_{g^{\otimes n}}\left(\frac{1}{n}\sum_{i=1}^{n} h(X_i)\frac{f(X_i)}{g(X_i)}\right) = \mathbb{E}_f(h(X))$$

is unbiased

Importance Sampling methods

$$\mathbb{V}_{g^{\otimes n}}\left[\frac{1}{n}\sum_{i=1}^{n} h(X_i)\frac{f(X_i)}{g(X_i)}\right] = \frac{1}{n}\mathbb{V}_g\left[h(X)\frac{f(X)}{g(X)}\right]$$

where

$$\mathbb{V}_g\left[h(X)\frac{f(X)}{g(X)}\right] = \mathbb{E}_f\left[h(X)^2\frac{f(X)}{g(X)}\right] - [\mathbb{E}_f(h(X))]^2$$

$$\frac{1}{n}\left[\frac{1}{n-1}\sum_{i=1}^{n}\left(h(X_i)\frac{f(X_i)}{g(X_i)} - \frac{1}{n}\sum_{j=1}^{n} h(X_j)\frac{f(X_j)}{g(X_j)}\right)^2\right]$$

is an unbiased estimator of $\mathbb{V}_g\left[h(X)\frac{f(X)}{g(X)}\right]/n$

Importance Sampling methods

The importance function that minimise $\mathbb{V}_g \left[h(X) \frac{f(X)}{g(X)} \right]$ is

$$g^*(x) = \frac{f(x)|h(x)|}{\int f(x)|h(x)|d\mu(x)}$$

$f|h|$ is absolutely continuous with respect to g^*

Importance Sampling methods

If $\mathbb{E}_g\left[\left|h(X)\frac{f(X)}{g(X)}\right|^2\right] = \mathbb{E}_f\left[|h(X)|^2\,\frac{f(X)}{g(X)}\right] < \infty$

$$\frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} h(X_i)\frac{f(X_i)}{g(X_i)} - \mathbb{E}_f(h(X))\right)}{\sqrt{\mathbb{V}_g\left[h(X)f(X)/g(X)\right]}} \longrightarrow_{\mathscr{L}} \mathrm{N}(0,1)$$

If $f(x)/g(x) < M$ and $\mathbb{V}_f(h(X)) < \infty$

$$\mathbb{E}_f\left[|h(X)|^2\,\frac{f(X)}{g(X)}\right] < \infty$$

Importance Sampling methods

There are many cases where the normalization constant of f is unknown (Bayesian statistic)

$$f(x) = \tilde{f}(x) \bigg/ \int \tilde{f}(x) d\mu(x) = \tilde{f}(x)/c$$

Self-normalized importance sampling estimator of $\mathbb{E}_f(h(X))$

$$\sum_{i=1}^{n} h(X_i) \frac{f(X_i)}{g(X_i)} \bigg/ \sum_{i=1}^{n} \frac{f(X_i)}{g(X_i)}$$

where $X_1, \ldots, X_n$ **is an iid sample from** g

Importance Sampling methods

If f is absolutely continuous with respect to g,

$$\sum_{i=1}^{n} h(X_i)\frac{f(X_i)}{g(X_i)} \Big/ \sum_{i=1}^{n} \frac{f(X_i)}{g(X_i)} \longrightarrow_{\text{ps}} \mathbb{E}_f(h(X))$$

is convergent

$$\mathbb{E}_{g^{\otimes n}}\left(\sum_{i=1}^{n} h(X_i)\frac{f(X_i)}{g(X_i)} \Big/ \sum_{i=1}^{n} \frac{f(X_i)}{g(X_i)}\right) \neq \mathbb{E}_f(h(X))$$

Importance Sampling methods

If $\mathbb{E}_f\left[|h(X)|^2 \frac{f(X)}{g(X)}\right] < \infty$, $\mathbb{E}_f\left[\frac{f(X)}{g(X)}\right] < \infty$,

$$\sqrt{n}\left(\sum_{i=1}^{n} h(X_i)\frac{f(X_i)}{g(X_i)} \Big/ \sum_{i=1}^{n}\frac{f(X_i)}{g(X_i)} - \mathbb{E}_f(h(X))\right) \longrightarrow_{\mathscr{L}}$$

$$\mathcal{N}\left(0, \mathbb{E}_f\left([h(X)-\mathbb{E}_f(h(X))]^2 f(X)/g(X)\right)\right)$$

The importance function that minimise $\mathbb{E}_f\left([h(X)-\mathbb{E}_f(h(X))]^2 f(X)/g(X)\right)$ is

$$g^{\#}(x) = \frac{f(x)|h(x)-\mathbb{E}_f(h(X))|}{\int f(x)|h(x)-\mathbb{E}_f(h(X))|d\mu(x)}\,.$$

Reminders and Additions on Markov Chains

Definition
A Markov chain is a random process $(X_k)_{k\in\mathbb{N}}$ such that

$$\mathbb{P}(X_k \in A|X_0 = x_0, \ldots, X_{k-1} = x_{k-1}) =$$

$$\mathbb{P}(X_k \in A|X_{k-1} = x_{k-1})$$

The Markov chain is homogenous if $\mathbb{P}(X_k \in A|X_{k-1} = x)$ does not depend on k

Example: random walk

$(X_k)_{k\in\mathbb{N}}$ such that

$$X_0 \sim \nu$$

and

$$X_k = X_{k-1} + \varepsilon_k, \quad \forall k \in \mathbb{N}^*$$

where $\varepsilon_1, \ldots$ is a random process with iid variables and probability distribution $\mathscr{L}$

Reminders and Additions on Markov Chains

Definition A (transition) kernel on $(\Omega, \mathscr{A})$ is an application $P : (\Omega, \mathscr{A}) \longrightarrow [0, 1]$ such that

1) $\forall A \in \mathscr{A}$, $P(\cdot, A)$ is measurable
2) $\forall x \in \Omega$, $P(x, \cdot)$ is a probability distribution on $(\Omega, \mathscr{A})$

$(X_k)_{k\in\mathbb{N}}$ is an homogenous Markov chain with kernel P if

$$\mathbb{P}(X_k \in A | X_{k-1} = x) = P(x, A), \quad \forall x \in \Omega, \quad \forall A \in \mathscr{A}\,.$$

Reminders and Additions on Markov Chains

For the random walk if $\mathscr{L} = \mathrm{N}(0, \sigma^2)$, $(X_k)_{k\in\mathbb{N}}$ is an homogenous Markov chain with kernel

$$P(x, A) = \int_A \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y-x)^2\right) dy$$

Reminders and Additions on Markov Chains

Let $(X_k)_{k\in\mathbb{N}}$ be an homogenous Markov chain with kernel P and initial distribution $X_0 \sim \nu$, we note

- P_ν the distribution of the chain $(X_k)_{k\in\mathbb{N}}$
- νP^k the distribution of X_k : $\forall A \in \mathscr{A}$,

$$\nu P^k(A) = \mathbb{P}(X_k \in A)$$

- $P^k(x, A) = \mathbb{P}(X_k \in A | X_0 = x)$

Reminders and Additions on Markov Chains

Let Π be a probability distribution on $(\Omega, \mathscr{A})$

We can simulate Π in an approximate way using a homogeneous Markov chain

To do this, one must be able to build a P kernel such that for any initial ν, $\nu P^k \longrightarrow_{VT} \Pi$

Total variation convergence

$$\|\nu P^k - \pi\|_{VT} = \sup_{A \in \mathscr{A}} |\nu P^k(A) - \Pi(A)|$$

Typically

$$\lim_{k \longrightarrow \infty} \nu P^k(A) = \Pi(A)$$

Reminders and Additions on Markov Chains

Definition

- P is Π-irreducible if $\forall x \in \Omega$ and $\forall A \in \mathscr{A}$ such that $\Pi(A) > 0$, $\exists k (= k(x, A))$ tel que $P^k(x, A) > 0$

- P is Π-invariant iff $\Pi P = \Pi$

$$\Pi P(A) = \int \Pi(dx_0) P(x_0, A) = \int_A \Pi(dx)$$

- P is Π-reversible iff $\forall A, B \in \mathscr{A}$,

$$\int_A P(x, B) \Pi(dx) = \int_B P(x, A) \Pi(dx)$$

Reminders and Additions on Markov Chains

If P is Π-reversible then P is Π-invariant

Indeed if P is Π-reversible, $\forall B \in \mathscr{A}$,

$$\int_\Omega P(x, B)\Pi(dx) = \int_B P(x, \Omega)\Pi(dx) = \int_B \Pi(dx)$$

Definition

- P is periodic with period $d \geqslant 2$ if there exists a partition $\Omega_1, \ldots, \Omega_d$ de Ω such that $\forall x \in \Omega_i$, $P(x, \Omega_{i+1}) = 1$, $\forall i$ with the convention $d + 1 = 1$
- A chain Π-irreducible and Π-invariant is recurrent if $\forall A \in \mathscr{A}$ such that $\pi(A) > 0$
  1) $\forall x \in \Omega$, $\mathbb{P}(X_k \in A$ infinitely often$|X_0 = x) > 0$
  2) $\exists x \in \Omega$, $\mathbb{P}(X_k \in A$ infinitely often$|X_0 = x) = 1$
- The chain is Harris-recurrent is 2) is verified for all $x \in \Omega$
- The chain is ergodic if it is Harris-recurrent and aperiodic

Convergence of Markov chains

If P is Π-irreducible and Π-invariant then P is recurrent

In that case, the invariant measure is unique (up to a multiplicative constant)

The chain is said to be positive recurrent if the invariant measure is a probability distribution

Convergence of Markov chains

Theorem Suppose that P is Π-irreducible et Π-invariant, then P is positive recurrent and Π is the unique invariant distribution of P. If P is Harris-recurrent and aperiodic (ergodic) then

$$\nu \mathrm{P}^k \longrightarrow_{VT} \Pi$$

The Harris-recurrence condition is difficult to obtain

It is satisfied for two main families of simulators: the Gibbs sampler and the Metropolis-Hastings algorithm

Convergence of Markov chains

Theorem If the Markov chain $(X_k)_{k\in\mathbb{N}}$ is ergodic with stationary distribution Π and if h is a real function such that$\mathbb{E}_\Pi(|h(X)|) < \infty$, then, whatever the initial distribution ν,

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i) \longrightarrow_{\text{ps}} \mathbb{E}_\Pi(h(X))$$

Convergence speed?

Convergence of Markov chains

Definition The Markov chain $(X_k)_{k\in\mathbb{N}}$ with kernel P is said to be uniformly ergodic if there is $M > 0$ and $0 < r < 1$ such that

$$\sup_{x\in\Omega}\sup_{A\in\mathscr{A}} |P^n(x, A) - \Pi(A)| \leqslant Mr^n$$

Theorem If the Markov chain $(X_k)_{k\in\mathbb{N}}$ is uniformly ergodic with stationary distribution Π and if h such that $\mathbb{E}_\Pi(|h(X)|) < \infty$ then, whatever the initial distribution ν, there is $\sigma(h) > 0$ such that

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} h(X_i) - \mathbb{E}_\Pi(h(X))\right) \longrightarrow_{\mathscr{L}} N(0, (\sigma(h))^2)$$

The Metropolis-Hastings algorithm

Target distribution

$$\Pi(dx) = \pi(x)\mu(dx)$$

Kernel Q for x such that $\pi(x) > 0$

$$Q(x, dy) = q(x, y)\mu(dy)$$

The Metropolis-Hastings algorithm

Choose $x^{(0)}$ such that $\pi\left(x^{(0)}\right) > 0$ and set $t = 1$

$(*)$ Generate $\tilde{x} \sim Q(x^{(t-1)}, \cdot)$

If $\pi(\tilde{x}) = 0$ then set $x^{(t)} = x^{(t-1)}$, $t = t + 1$ and return to $(*)$

If $\pi(\tilde{x}) > 0$ calculate

$$\rho(x^{(t-1)}, \tilde{x}) = \frac{\pi(\tilde{x}) \big/ q(x^{(t-1)}, \tilde{x})}{\pi(x^{(t-1)}) \big/ q(\tilde{x}, x^{(t-1)})}$$

Generate $u \sim \mathscr{U}([0, 1])$

If $u \leqslant \rho(x^{(t-1)}, \tilde{x})$ then $x^{(t)} = \tilde{x}$ else $x^{(t)} = x^{(t-1)}$

set $t = t + 1$ and return to $(*)$

The Metropolis-Hastings algorithm

Starting from x ($\pi(x) > 0$), the acceptance probability of y ($\pi(y) > 0$) is given by

$$\alpha(x, y) = \min\left[1, \frac{\pi(y)/q(x, y)}{\pi(x)/q(y, x)}\right]$$

Whatever the value of x such as $\pi(x) > 0$, the kernel associated with the Metropolis-Hastings algorithm is given by

$$K(x, dy) = q(x, y)\mu(dy)\alpha(x, y) + \left[1 - \int q(x, z)\alpha(x, z)\mu(dz)\right]\delta_x(dy)$$

where $\delta_x(\cdot)$ is the Dirac mass at point x

The Metropolis-Hastings algorithm

We can easily show that K is Π-reversible

Indeed

$$\Pi(dx)K(x,dy) = \min\left[\pi(y)q(y,x), \pi(x)q(x,y)\right]\mu(dy)\mu(dx)$$

$$+\left\{\pi(x)\mu(dx) - \int \min\left[\pi(z)q(z,x), \pi(x)q(x,z)\right]\mu(dz)\right\}\delta_x(dy)$$

and

$$\Pi(dy)K(y,dx) = \min\left[\pi(x)q(x,y), \pi(y)q(y,x)\right]\mu(dx)\mu(dy)$$

$$+\left\{\pi(y)\mu(dy) - \int \min\left[\pi(x)q(x,z), \pi(z)q(z,x)\right]\mu(dz)\right\}\delta_y(dx)$$

The Metropolis-Hastings algorithm

Theorem If the kernel Q is π-irreducible, the Markov chain generated with the Metropolis-Hastings algorithm is π-irreducible, π-invariant, Harris-recurrent and aperiodic

Two particular cases

- Q is a random walk kernel: $q(x,y) = q_{RW}(x-y)$ and $q_{RW}(x) = q_{RW}(-x)$
- Q is an independent kernel: $q(x,y) = q(y)$

The Gibbs sampler

Goal: generate simulations from multivariate distributions

Let $X = (X_1, X_2, \dots, X_d)$ with probability distribution Π

Note Π_i the conditional distribution of X_i given
$X_{-i} = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_d) = x_{-i}$

Π_i **is called the full conditional distribution of** X_i

The Gibbs sampler

$$
\begin{array}{ll}
 & \text{Choose } x^{(0)} \text{ and set } t = 1 \\
(*) & \text{Generate } x_1^{(t)} \sim \Pi_1(\cdot|x_2^{(t-1)}, \ldots, x_d^{(t-1)}) \\
 & \text{Generate } x_2^{(t)} \sim \Pi_2(\cdot|x_1^{(t)}, x_3^{(t-1)}, \ldots, x_d^{(t-1)}) \\
 & \text{Generate } x_3^{(t)} \sim \Pi_3(\cdot|x_1^{(t)}, x_2^{(t)}, x_4^{(t-1)} \ldots, x_d^{(t-1)}) \\
 & \ldots \\
 & \text{Generate } x_d^{(t)} \sim \Pi_d(\cdot|x_1^{(t)}, \ldots, x_{d-1}^{(t)}) \\
 & \text{Set } t = t + 1 \text{ and return to } (*)
\end{array}
$$

Theorem The Markov chain generated using the Gibbs sampler is Π-irreducible, Π-invariant, Harris-recurrent and aperiodic