QFT, solutions to problem sheet 4

Problem 1: Asymptotic overlap between wave packets

With $a^\dagger_\chi(t) = \int \mathrm{d}^3k\, \chi(\vec{k}) a^\dagger(\vec{k}, t)$*, show that*

$$\lim_{t\to\pm\infty} \langle\psi|a^\dagger_\chi(t)|0\rangle = 0\,,$$

i.e. the overlap between a one-particle and an n-particle wave packet tends to zero at $t \to \pm\infty$*.*

We have

$$a^\dagger(\vec{k}, t) = -i \int \mathrm{d}^3x\, e^{-ikx} \overleftrightarrow{\partial_0} \phi(x)$$

and therefore

$$\begin{aligned}
&\lim_{t\to\pm\infty} \langle\psi|a^\dagger_\chi(t)|0\rangle \\
&\quad= \lim_{t\to\pm\infty} \int \mathrm{d}^3p \int \mathrm{d}^3k\; \psi^*(\vec{p})\chi(\vec{k})\langle\lambda_p|a^\dagger(\vec{k},t)|0\rangle \\
&\quad= \lim_{t\to\pm\infty} (-i) \int \mathrm{d}^3p \int \mathrm{d}^3k \int \mathrm{d}^3x\; \psi^*(\vec{p})\chi(\vec{k}) e^{-ikx} \overleftrightarrow{\partial_0} \langle\lambda_p|\phi(x)|0\rangle \\
&\quad= \lim_{t\to\pm\infty} (-i) \int \mathrm{d}^3p \int \mathrm{d}^3k \int \mathrm{d}^3x\; \psi^*(\vec{p})\chi(\vec{k}) e^{-ikx} \overleftrightarrow{\partial_0} \underbrace{\langle\lambda_p|e^{iPx}}_{=e^{ipx}\langle\lambda_p|} \phi(0) \underbrace{e^{-iPx}|0\rangle}_{=|0\rangle} \\
&\quad= \lim_{t\to\pm\infty} \int \mathrm{d}^3p \int \mathrm{d}^3k \int \mathrm{d}^3x\; \psi^*(\vec{p})\chi(\vec{k})(k^0+p^0) e^{i(p^0-k^0)t} e^{-i(\vec{p}-\vec{k})\cdot\vec{x}} \langle\lambda_p|\phi(0)|0\rangle \\
&\quad= \lim_{t\to\pm\infty} \int \mathrm{d}^3p \int \mathrm{d}^3k \int \mathrm{d}^3x\; \psi^*(\vec{p})\chi(\vec{k})(k^0+p^0) e^{i(p^0-k^0)t} (2\pi)^3\delta^{(3)}(\vec{p}-\vec{k}) \langle\lambda_p|\phi(0)|0\rangle \\
&\quad= \lim_{t\to\pm\infty} (2\pi)^3 \int \mathrm{d}^3p\; \psi^*(\vec{p})\chi(\vec{p}) \left(\sqrt{\vec{p}^2+M^2} + \sqrt{\vec{p}^2+m^2}\right) \\
&\qquad\qquad \times \exp\left(i\left(\sqrt{\vec{p}^2+M^2} - \sqrt{\vec{p}^2+m^2}\right)t\right) \langle\lambda_p|\phi(0)|0\rangle
\end{aligned}$$

In order to apply the stationary phase approximation, we need to show that the function $\varphi(\vec{p}) = \sqrt{\vec{p}^2+M^2} - \sqrt{\vec{p}^2+m^2}$ has a single and nondegenerate critical point. This is indeed the case since

$$\vec{\nabla}\varphi(\vec{p}) = \vec{p}\left(\frac{1}{\sqrt{\vec{p}^2+M^2}} - \frac{1}{\sqrt{\vec{p}^2+m^2}}\right)$$

is zero iff $\vec{p} = \vec{0}$, and since

$$H_\varphi(\vec{0}) = \left(\frac{1}{M} - \frac{1}{m}\right)\mathbb{1}$$

is nonsingular. From the stationary phase approximation, we can therefore conclude that the integral tends to zero as $|t|^{-3/2}$ asymptotically.

Problem 2: Gaussian and oscillatory integrals

1. *Let $a, b \in \mathbb{C}$ with $\mathrm{Re}\, a > 0$. Show that:*

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-\frac{1}{2}a\,x^2 + b\,x\right) = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{1}{2}\frac{b^2}{a}\right) .$$

Completing the square gives

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-\frac{1}{2}a\,x^2 + b\,x\right) = \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-\frac{1}{2}a \underbrace{\left(x - \frac{b}{a}\right)^2}_{\equiv y^2} + \frac{b^2}{2a}\right)$$

$$= \exp\left(\frac{b^2}{2a}\right) \int_{-\infty}^{\infty} \mathrm{d}y \, \exp\left(-\frac{1}{2}a\,y^2\right) = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{b^2}{2a}\right) .$$

2. *Let $a \in \mathbb{R}_+^*$ and $b \in \mathbb{C}$. We define the integral*

$$I = \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(\frac{i}{2}a\,x^2 + i\,b\,x\right)$$

by a complex line integral whose curve of integration $\mathcal{C}_{X,\epsilon}$ is the straight line connecting the points $-X(1+i\epsilon)$ and $+X(1+i\epsilon)$:

$$I = \lim_{\epsilon \searrow 0} \lim_{X \to \infty} \int_{\mathcal{C}_{X,\epsilon}} \mathrm{d}z \, \exp\left(\frac{i}{2}a\,z^2 + i\,b\,z\right) .$$

Calculate I. Compare your result with the stationary phase approximation given in exercise 1.

We parameterize the curve $\mathcal{C}_{X,\epsilon}$ by setting $z = t(1+i\epsilon)$ and taking the real parameter t between $-X$ and $+X$:

$$I = \lim_{\epsilon \searrow 0} \lim_{X \to \infty} \int_{-X}^{X} \mathrm{d}t \, (1+i\epsilon) \exp\left(\frac{i}{2}\,a\,t^2(1+i\epsilon)^2 + i\,b\,t\,(1+i\epsilon)\right)$$

The real part of the coefficient of t^2 in the exponential is $-a\epsilon < 0$, hence we can use the result of 1. to find

$$I = \lim_{\epsilon \searrow 0} (1+i\epsilon) \sqrt{\frac{2\pi i}{a(1+i\epsilon)^2}} \exp\left(-\frac{i\,b^2}{2a}\right) = \sqrt{\frac{2\pi i}{a}} \exp\left(-\frac{i\,b^2}{2a}\right) .$$

Setting $n = 1$, $\lambda = 1$ and $\varphi(x) = \frac{a}{2}x^2 + bx$, one finds that the stationary phase approximation is exact at leading order for this integral, i.e. the $\mathcal{O}(\lambda^{-n/2})$ term given explicitly in Problem 1 is already the complete result.

3. *Let A be a real symmetric positive definite $n \times n$ matrix. Show that*

$$\int \mathrm{d}^n x \exp\left(-\frac{1}{2}x^T A x\right) = \sqrt{\frac{(2\pi)^n}{\det A}}\,, \qquad \int \mathrm{d}^n x \exp\left(\frac{i}{2}x^T A x\right) = e^{in\pi/4}\sqrt{\frac{(2\pi)^n}{\det A}}.$$

As a symmetric matrix, A is diagonalized by an orthogonal matrix O such that $O^T O = \mathbb{1}$ and $OAO^T = \mathrm{diag}\,(a_1, a_2, \ldots, a_n)$. We have

$$\int \mathrm{d}^n x \exp\left(-\frac{1}{2} x^T A x\right) = \int \mathrm{d}^n x \exp\left(-\frac{1}{2} \underbrace{x^T O^T}_{=y^T} OAO^T \underbrace{Ox}_{=y}\right) .$$

Since O is orthogonal, the Jacobian for the change of integration variables $x \to y$ is $\det O = 1$ and thus $\mathrm{d}^n x = \mathrm{d}^n y$. In the y basis, the integral factorizes:

$$\begin{aligned}\int \mathrm{d}^n x \exp\left(-\frac{1}{2} x^T A x\right) &= \int \mathrm{d}^n y \exp\left(-\frac{1}{2} \sum_{i=1}^n a_i\, y_i^2\right) = \prod_{i=1}^n \int \mathrm{d}y_i \exp\left(-\frac{1}{2} a_i\, y_i^2\right) \\ &= \prod_{i=1}^n \sqrt{\frac{2\pi}{a_i}} = \sqrt{\frac{(2\pi)^n}{\det A}} .\end{aligned}$$

The second formula follows from this result and from part 2., taking into account that every integral contributes a factor of $\sqrt{i} = e^{i\pi/4}$.