# TD Compléments en régression linéaire multiple



## Exercice 1

$y_i = b x_i + \varepsilon_i$, $\mathbb{E}(\varepsilon_i) = 0$, $\mathbb{V}(\varepsilon_i) = \sigma^2$

$\mathrm{Cov}(\varepsilon_i, \varepsilon_j) = 0$

$$\hat{b} \in \arg\min_{b \in \mathbb{R}} \sum_{i=1}^{n} (y_i - b x_i)^2$$

$$\hat{b} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} \qquad \mathbb{E}(\hat{b}) = b$$

$$\mathbb{V}(\hat{b}) = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$

$$\tilde{b} = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i} \qquad \mathbb{E}(\tilde{b}) = b$$

$$\mathbb{V}(\tilde{b}) = \frac{n\sigma^2}{\left(\sum_{i=1}^{n} x_i\right)^2}$$

$\left(\sum x_i\right)^2 \leq n \sum x_i^2$ (Cauchy-Schwarz)

$\langle u, v \rangle^2 \leq \|u\|^2 \|v\|^2$

$$\Rightarrow \frac{\sigma^2}{\sum x_i^2} \leq \frac{n\sigma^2}{(\sum x_i)^2}$$

$$\Rightarrow V(\hat{\beta}) \leq V(\tilde{\beta})$$

On retrouve le théorème de Gauss-Markov.

## Exercice 2

1) $n = 5$ , $x = (x_1, \dots, x_n)$

$z = (z_1, \dots, z_n)$

$$C^e(x,z) = \left(\frac{1}{n-1}\right) \sum_{i=1}^{n} (x_i - \bar{x})(z_i - \bar{z})$$

$$C^e(x,z) = \left(\frac{1}{n-1}\right) \sum_{i=1}^{n} x_i z_i - \left(\frac{n}{n-1}\right) \bar{x}\bar{z}$$

$$C^e(x,z) = \frac{1}{4}$$

$$V^e(x) = \left(\frac{1}{n-1}\right) \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$V^e(x) = \left(\frac{1}{n-1}\right) \sum_{i=1}^{n} x_i^2 - \left(\frac{n}{n-1}\right)(\bar{x})^2$$

$$\mathbb{V}^e(x) = \frac{3}{4} - \frac{9}{20} = \frac{6}{20} = \frac{3}{10}$$

$$\mathbb{V}^e(z) = \frac{1}{4}$$

$$\text{cor}^e(x,z) = \frac{1/4}{\sqrt{1/4}\sqrt{3/10}} = \sqrt{\frac{5}{6}}$$

$$\# 0.91$$

2] $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2) = (-1, 3, 4)$

Droite de régression pour le nuage de points

$$\bar{y} = -1 + 3\bar{x} + 4\bar{z}$$

$$\Rightarrow \bar{y} = 4/5$$

3] $\|\hat{y}\|^2 = \langle -\Pi_5 + 3x + 4z, -\Pi_5 + 3x + 4z \rangle$

$= 54$ $(= \|X(X^TX)^{-1}X^Ty\|^2 = y^TX(X^TX)^{-1}X^Ty)$

$$\|\hat{y} - \bar{y}\Pi_5\|^2 = \|\hat{y}\|^2 - \|\bar{y}\Pi_5\|$$

$$= 54 - 16/5$$

(4)

$\Leftrightarrow TSS = RSS + ESS$

$\Leftrightarrow TSS = 3 + 54 - \frac{16}{5}$

$\Leftrightarrow TSS = 57 - \frac{16}{5} = \frac{269}{5}$

$R^2 = \frac{ESS}{TSS} = \frac{254}{269} \# 0,94$

4) $X_0^T y = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$

$\Rightarrow \hat{\beta}_0 = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}^{-1} \begin{pmatrix} 4 \\ 10 \end{pmatrix}$

$\Rightarrow \hat{\beta}_0 = \left(\frac{1}{6}\right) \begin{pmatrix} 3 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 10 \end{pmatrix}$

$\Rightarrow \hat{\beta}_0 = \begin{pmatrix} -3 \\ 19/3 \end{pmatrix}$

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$$\|\hat{y}_0\|^2 = \hat{\beta}_0^T X_0^T y = \frac{154}{3}$$

$$\|\hat{m}\|^2 + \|\hat{y}\|^2 = \|y\|^2$$

$$\|\hat{m}_0\|^2 + \|\hat{y}_0\|^2 = \|y\|^2$$

Par projection, ainsi

$$\|\hat{m}_0\|^2 + \|\hat{y}_0\|^2 = \|\hat{m}\|^2 + \|\hat{y}\|^2$$

$$\Rightarrow \|\hat{m}_0\|^2 = \|\hat{m}\|^2 + \|\hat{y}\|^2 - \|\hat{y}_0\|^2$$

$$= 3 + 54 - \frac{154}{3}$$

$$= \frac{17}{3}$$

$$RSS = 3 \quad < \quad RSS_0 = \frac{17}{3}$$

## Exercice 3

$$\hat{\beta}_0^{MCO} = \sum y_i x_i / \sum x_i^2$$

$$\hat{\alpha}^{GLS} = \sum \frac{w_i}{\sigma_i^2} y_i \Big/ \sum \frac{w_i^2}{\sigma_i^2}$$

$$V(\hat{\alpha}^{OLS}) = \frac{\sum w_i^2 \sigma_i^2}{(\sum w_i^2)^2}$$

$$V(\hat{\alpha}^{GLS}) = \frac{1}{\sum \frac{w_i^2}{\sigma_i^2}}$$

Diseguaglianza Cauchy - Schwarz

$$(\sum w_i^2)^2 \leq \sum \frac{w_i^2}{\sigma_i^2} \sum w_i^2 \sigma_i^2$$

$$\Rightarrow V(\hat{\alpha}^{GLS}) \leq V(\hat{\alpha}^{OLS})$$

Esercizio 4

$$\hat{\mu} = \sum \frac{\bar{y}_{i\cdot}}{\sigma_i^2} \Big/ \sum \frac{1}{\sigma_i^2}$$

dove $\bar{y}_{i\cdot} = \frac{1}{J} \sum_{j=1}^{J} y_{ij}$