QFT, solutions to problem sheet 12

Problem 1: The Grassmann algebra

1. Find the vector space dimension of the Grassmann algebra generated by $\mathbb{1}$ and $\{\theta_i\}_{i=1\ldots n}$.

The basis vectors can be taken to be $\mathbb{1}$ and products $\theta_{i_1}\cdots\theta_{i_k}$ of Grassmann generators; how many of the latter are independent?

- $\mathbb{1}$ is independent $\Rightarrow$ 1 dimension
- All θ_i are independent $\Rightarrow$ n more dimensions
- Among the n^2 products of the form $\theta_i\theta_j$, only $n(n-1)/2$ are independent $\Rightarrow$ $n(n-1)/2$ more dimensions
- Among the n^3 products of the form $\theta_i\theta_j\theta_k$, only $n(n-1)(n-2)/3!$ are independent $\Rightarrow$ $n(n-1)(n-2)/3!$ more dimensions
- ...
- Among the n^k products of the form $\theta_{i_1}\cdots\theta_{i_k}$, only $\frac{n!}{(n-k)!k!}$ are independent $\Rightarrow$ $\left(\begin{array}{c} n \\ k \end{array}\right)$ more dimensions
- ...
- There is only one independent product of n factors of θs, containing each θ exactly once. There are no nonzero products of more than n θs.

The overall dimension is

$$\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right) = 2^n\,.$$

2. Let $U=(U_{ij})$ be a unitary matrix of commuting numbers $(i,j=1\ldots n)$. Show that the Grassmann integral

$$\int \mathrm{d}^n\theta^*\,\mathrm{d}^n\theta\, f(\theta_i,\theta_i^*)$$

is invariant under the change of variables $\theta_i \to U_{ij}\theta_j$, $\theta_i^* \to U_{ij}^*\theta_j^*$.

The most general function of n Grassmann variables and their conjugates can be expanded as

$$f(\theta_i,\theta_i^*) = c + c_1\theta_1 + c_2\theta_2 + c^1\theta_1^* + \ldots + c_{12}\theta_1\theta_2 + c_1^1\theta_1\theta_1^* + \ldots + c_{1\ldots n}^{1\ldots n}\,\theta_1\theta_1^*\theta_2\theta_2^*\ldots\theta_n\theta_n^*$$

where the cs are complex coefficients. According to the Grassmann integration rules, only the last term contributes to the integral, whose value is

$$\int \mathrm{d}^n\theta^* \mathrm{d}^n\theta \; f(\theta_i, \theta_i^*) = c^{1\ldots n}_{1\ldots n} \equiv C\,.$$

We now note that

$$\theta_1\theta_2\ldots\theta_n = \frac{1}{n!}\epsilon^{i_1 i_2\ldots i_n}\theta_{i_1}\theta_{i_2}\ldots\theta_{i_n}$$

and that

$$\theta_{i_1}\theta_{i_2}\ldots\theta_{i_n} = \epsilon_{i_1 i_2\ldots i_n}\theta_1\theta_2\ldots\theta_n\,.$$

This allows to write, for $\theta'_i = U_{ij}\theta_j$,

$$\begin{aligned}\theta'_1\theta'_2\ldots\theta'_n &= \frac{1}{n!}\epsilon^{i_1 i_2\ldots i_n}\theta'_{i_1}\theta'_{i_2}\ldots\theta'_{i_n} = \frac{1}{n!}\epsilon^{i_1 i_2\ldots i_n}U_{i_1 j_1}\theta_{j_1}U_{i_2 j_2}\theta_{j_2}\ldots U_{i_n j_n}\theta_{j_n}\\ &= \frac{1}{n!}\epsilon^{i_1 i_2\ldots i_n}\epsilon^{j_1 j_2\ldots j_n}U_{i_1 j_1}\ldots U_{i_n j_n}\theta_1\theta_2\ldots\theta_n = (\det U)\theta_1\theta_2\ldots\theta_n\,.\end{aligned}$$

Similarly, one shows that

$$\theta_1^{*\prime}\theta_2^{*\prime}\ldots\theta_n^{*\prime} = (\det U)^*\theta_1^*\theta_2^*\ldots\theta_n^*$$

and therefore

$$\int \mathrm{d}^n\theta^* \mathrm{d}^n\theta \; f(U_{ij}\theta_j, U^*_{ij}\theta^*_j) = C|\det U|^2 = C = \int \mathrm{d}^n\theta^* \mathrm{d}^n\theta \; f(\theta_i, \theta_i^*)\,.$$

3. Let A be a hermitian $n \times n$ matrix of commuting numbers. Show that

$$\int \mathrm{d}^n\theta^* \,\mathrm{d}^n\theta \, e^{-\theta^\dagger A\,\theta} = \det A\,.$$

Choose U such that $U^\dagger A U = \mathrm{diag}\,(a_1, \ldots, a_n)$:

$$\int \mathrm{d}^n\theta^* \mathrm{d}^n\theta \; e^{-\theta^\dagger A\theta} = \int \mathrm{d}^n\theta^* \mathrm{d}^n\theta \; e^{-\sum_i \theta_i^* a_i \theta_i} = \prod_i a_i = \det A\,.$$

Problem 2: Yukawa theory

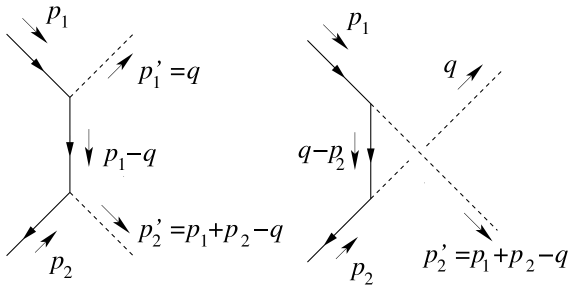
Consider the Yukawa Lagrangian

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - M)\psi + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - y\,\phi\overline{\psi}\psi,$$

where ψ is a Dirac field, ϕ is a real scalar field, and y is a coupling.

1. Compute the matrix element $\mathcal{M}_{\mathrm{fi}}$ for $\overline{\psi}\psi \to \phi\phi$ scattering to lowest order in perturbation theory.

At the tree level there are two diagrams which contribute:



To calculate the matrix element, we should amputate the external propagators, replacing the external (anti)fermion propagators with the respective basis spinors. This gives (with s_i the spin of the incident particle of momentum To calculate the matrix element, we should amputate the external propagators, replacing the external (anti)fermion propagators with the respective basis spinors. This gives (with s_i the spin of the incident particle of momentum p_i)

$$\begin{aligned}\mathcal{M}_{\text{fi}} &= \bar{v}_{s_2}(\vec{p}_2)(-iy)\frac{i(\not{p}_1 - \not{q} + M)}{(p_1 - q)^2 - M^2}(-iy)u_{s_1}(\vec{p}_1) + \bar{v}_{s_2}(\vec{p}_2)(-iy)\frac{i(\not{q} - \not{p}_2 + M)}{(p_2 - q)^2 - M^2}(-iy)u_{s_1}(\vec{p}_1) \\ &= -iy^2\bar{v}_{s_2}(\vec{p}_2)\left(\frac{\not{p}_1 - \not{q} + M}{m^2 - 2p_1 q} + \frac{\not{q} - \not{p}_2 + M}{m^2 - 2p_2 q}\right)u_{s_1}(\vec{p}_1)\end{aligned}$$

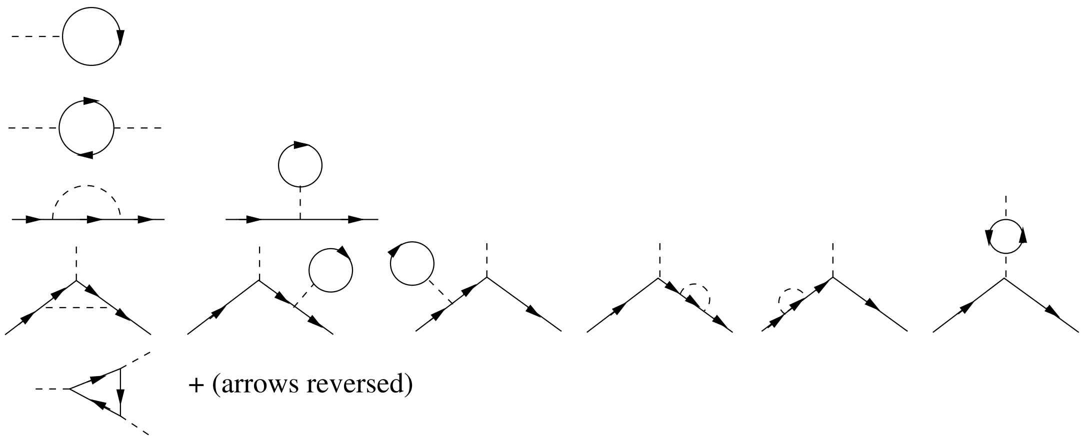
where in the last line we have used that the external particles are on shell, hence $p_1^2 = p_2^2 = M^2$ and $q^2 = m^2$.

(If you are motivated: Use this result to further calculate the total cross section as a function of the Mandelstam variables s, t, and u, assuming that the incoming particles are unpolarized — but be warned, this calculation is long and tedious, and best done with the help of a computer algebra system.)

Some steps for this calculation are detailed and the result is given in Mark Srednicki's QFT book (see chapters 45-48).

2. Draw the Feynman diagrams contributing to all the 1-point, 2-point, and 3-point *correlation functions*. Which of them do you expect to diverge? (There is no need to evaluate them in detail.)

The diagrams can be drawn as



where "+ (arrows reversed)" indicates that the same diagram(s) should be drawn again with all fermion arrows pointing in the other direction.

One can anticipate whether a diagram is UV-divergent by counting powers of loop momenta. In all of these diagrams, there is only one loop, and therefore only one momentum q to be integrated over. At large q, fermion loop propagators will scale as $\sim 1/q$ and scalar loop propagators will scale as $\sim 1/q^2$.

- The "tadpole" diagram of the first line therefore diverges as

$$\int^{\Lambda} \frac{d^4 q}{q} \sim \Lambda^3 .$$

  This indicates a divergent contribution to the scalar 1-point function, which should be subtracted by a counterterm. By momentum conservation, the momentum flowing into this diagram is zero, and therefore the contribution to the 1-point function is a momentum-independent constant. It is common to impose $\langle 0|\phi(x)|0\rangle = 0$ as a renormalization condition, so the counterterm will simply cancel the divergent tadpole graph, and the net effect is zero. This counterterm effectively removes the second diagram on the third line, and the second and third diagram on the fourth line.

- The diagram on the second line diverges as $\sim \Lambda^2$. This indicates the need for renormalizing the scalar propagator by introducing two counterterms for mass and wave function renormalization, just as in ϕ^4 theory. The sum of the loop diagram and the counterterm diagram will then be finite.

- The first diagram on the third line (the second one will cancel once the tadpole has been subtracted) could be expected to diverge as $\sim \Lambda$; in fact the leading divergence cancels for symmetry reasons, but the diagram still diverges as $\sim \log \Lambda$. This divergence can be absorbed in the wave function renormalization of ψ, upon introducing a suitable counterterm. Again, the sum of the loop diagram and the counterterm diagram will be finite.

- The first diagram on the fourth line (the second and third ones will cancel) diverges as $\sim \log \Lambda$. One therefore needs to introduce a counterterm for the Yukawa vertex.

- The last three diagrams on the fourth line diverge, because they contain divergent subdiagrams (the ones on the second and third line). Once the latter have been renormalized, they will be finite if the corresponding counterterm diagram (with the loop replaced by a counterterm insertion) is also added.

- The two diagrams on the last line diverge as $\sim \Lambda$. One therefore needs a counterterm for the scalar three-point function. Since a scalar cubic interaction is induced by loops, there is no reason not to include it at the tree level in the first place; the Lagrangian should therefore already include a term $\sim \phi^3$, and the Feynman rules should include a corresponding vertex.

3. Draw the *amputated* Feynman diagrams contributing to $\phi\phi \to \phi\phi$ and $\psi\psi \to \psi\psi$ scattering. Which of them do you expect to diverge?

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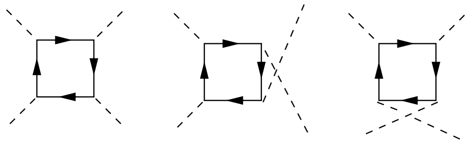

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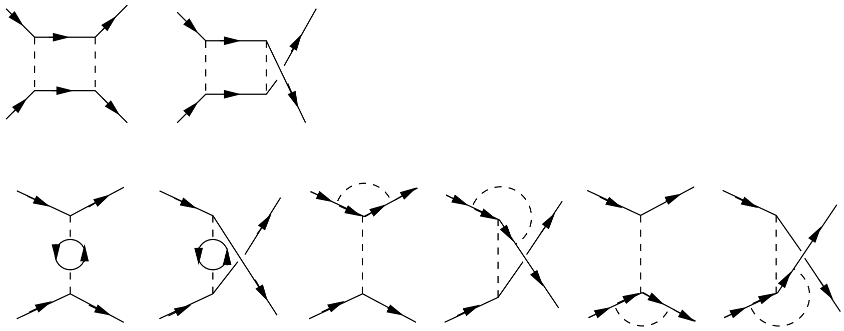

- The six diagrams on the first line diverge as $\sim \log \Lambda$. Similarly as for the three-point interaction, the tree-level Lagrangian should have included a term $\sim \phi^4$, since even if it is taken to be absent at the tree level, it will be generated by loop corrections.

- The diagrams on the second line are finite because the loop integral in the UV behaves as $\sim \int d^4q \, \frac{1}{q}\frac{1}{q}\frac{1}{q^2}\frac{1}{q^2}$, which is convergent. There is therefore no need to introduce a counterterm for a four-fermion interaction (this we could have guessed already because a $(\overline{\psi}\psi)^2$ term in $\mathcal{L}$ has dimension 6 and is therefore not renormalizable).

- The diagrams on the last line are actually divergent, despite contributing to the fermion four-point function. But this is because they contain divergent subdiagrams; once these has been renormalized, and the corresponding counterterm diagram added, they are finite as they should be.

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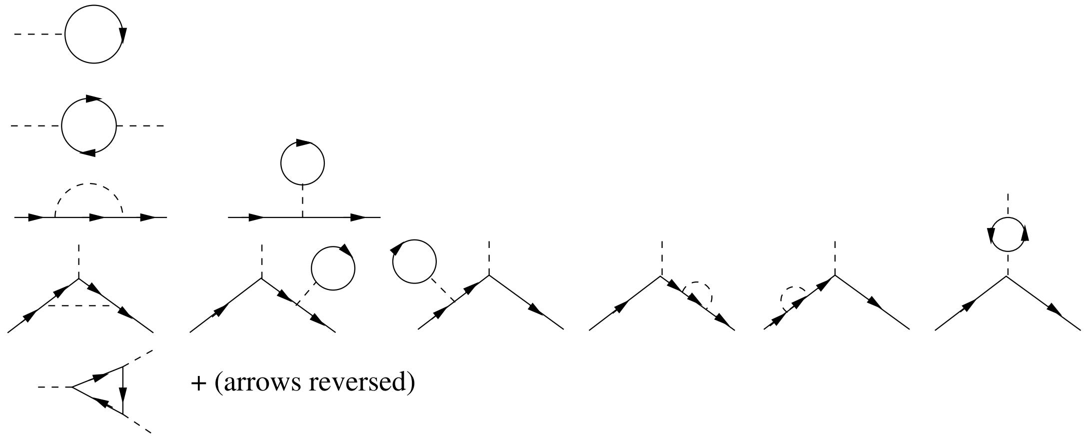

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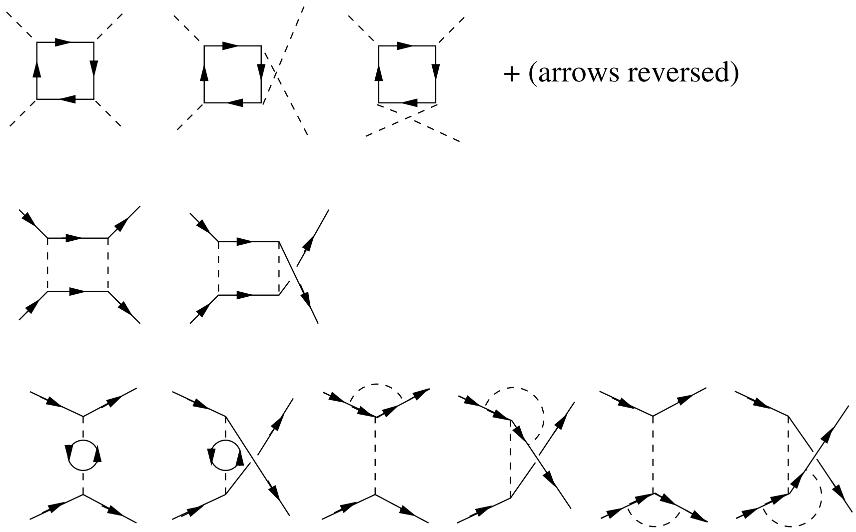

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