QFT, solutions to problem sheet 10

Problem 1: Compton scattering

Consider an $e\gamma \to e\gamma$ scattering process. The four-momenta in the initial state are p_1 for the electron and p_2 for the photon, while in the final state they are p_2' for the photon and $p_1' = p_1 + p_2 - p_2'$ for the electron. A tree-level calculation in quantum electrodynamics gives the squared matrix element

$$|\overline{\mathcal{M}}|^2 = 32\pi^2\,\alpha^2 \left(\frac{p_1 p_2'}{p_1 p_2} + \frac{p_1 p_2}{p_1 p_2'} + 2\,m^2 \left(\frac{1}{p_1 p_2} - \frac{1}{p_1 p_2'} \right) + m^4 \left(\frac{1}{p_1 p_2} - \frac{1}{p_1 p_2'} \right)^2 \right) .$$

Here α is the fine-structure constant, m is the electron mass, and the bar in $\overline{\mathcal{M}}$ indicates that we have averaged over initial spin and polarization states and summed over final ones.

Starting from this expression, derive the *Klein-Nishina formula*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{m^2}\frac{\omega'^2}{\omega^2}\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right) ,$$

where ω and ω' are the initial and final photon energies, and θ is the scattering angle between the two photons, in a frame where the initial electron is at rest.

According to the lecture, the differential cross-section for $2 \to 2$ scattering is given by

$$\mathrm{d}\sigma = \frac{1}{4\,E_1\,E_2}\frac{1}{|\vec{v}_1 - \vec{v}_2|}\frac{\mathrm{d}^3 p_1'}{2\,E_1'\,(2\pi)^3}\frac{\mathrm{d}^3 p_2'}{2\,E_2'\,(2\pi)^3}\,(2\pi)^4\,\delta^{(4)}\left(p_1 + p_2 - p_1' - p_2'\right)|\overline{\mathcal{M}}|^2 .$$

We begin by choosing a coordinate system: Initially the electron is at rest at the origin, the incoming photon is aligned with the z-direction, and the two photons lie in the (y, z)-plane. This gives

$$p_1 = (m, \vec{0}) , \quad p_2 = (\omega, \omega\vec{e}_z) , \quad p_1' = (E_1', \vec{p}_1{}') , \quad p_2' = (\omega', \omega'\sin\theta\vec{e}_y + \omega'\cos\theta\vec{e}_z)$$

where $E_1' = \sqrt{\vec{p}_1{}'^2 + m^2}$; here we have used that all particles are on shell and that the photon is massless. In this frame, $|\vec{v}_1 - \vec{v}_2| = 1$, and therefore

$$\mathrm{d}\sigma = \frac{1}{4\,\omega m}\frac{\mathrm{d}^3 p_1'}{2\,E_1'\,(2\pi)^3}\frac{\mathrm{d}^3 p_2'}{2\,\omega'\,(2\pi)^3}\,(2\pi)^4\,\delta^{(4)}\left(p_1 + p_2 - p_1' - p_2'\right)|\overline{\mathcal{M}}|^2 .$$

Splitting the four-dimensional delta function into an energy-conserving part and a 3-momentum conserving part,

$$\begin{aligned}&(2\pi)^4\,\delta^{(4)}(p_1 + p_2 - p_1' - p_2') \\ &\quad = (2\pi)\,\delta\left(m + \omega - E_1' - \omega'\right)\,(2\pi)^3\delta^{(3)}\left(\omega\vec{e}_z - \vec{p}_1{}' - \omega'\sin\theta\vec{e}_y - \omega'\cos\theta\vec{e}_z\right)\end{aligned}$$

we notice that the latter enforces

$$\vec{p}_1{}' = (\omega - \omega'\cos\theta)\vec{e}_z - \omega'\sin\theta\vec{e}_y , \quad E_1' = \sqrt{\omega^2 + \omega'^2 + m^2 - 2\omega\omega'\cos\theta} \qquad (1)$$

and therefore

$$\mathrm{d}\sigma = \frac{1}{8\,m\omega E_1'}\frac{\mathrm{d}^3 p_2'}{2\,\omega'\,(2\pi)^3}\,(2\pi)\,\delta\left(m+\omega-E_1'-\omega'\right)\,|\overline{\mathcal{M}}|^2$$

where E_1' is now a function of ω, ω' and θ given in (1). Transforming to polar coordinates and integrating over the angle ϕ gives

$$\mathrm{d}^3 p_2' = 2\pi\,\omega'^2 \mathrm{d}\omega'\,\mathrm{d}\cos\theta$$

and hence

$$\mathrm{d}\sigma = \frac{1}{32\pi}\frac{\omega'}{m\omega E_1'}\,\mathrm{d}\omega'\,\mathrm{d}\cos\theta\;\delta\left(m+\omega-E_1'-\omega'\right)\,|\overline{\mathcal{M}}|^2\,.$$

We should now transform the energy-conserving delta function, because its argument is a nontrivial function of the integration variable ω'. In general,

$$\delta(f(\omega')) = \sum_{\{\omega_0'\,:\,f(\omega_0')=0\}}\frac{1}{|\frac{\partial f}{\partial\omega'}(\omega_0')|}\,\delta(\omega'-\omega_0')\,.$$

Here, with $f(\omega') = m+\omega-E_1'(\omega')-\omega'$, we have

$$\left|\frac{\partial}{\partial\omega'}\left(m+\omega-E_1'-\omega'\right)\right| = \left|-1-\frac{\omega'-\omega\cos\theta}{E_1'}\right| = \left|\frac{E_1'+\omega'-\omega\cos\theta}{E_1'}\right| = \frac{m+\omega(1-\cos\theta)}{E_1'}$$

where the last equality holds only under the delta function. Therefore

$$\mathrm{d}\sigma = \frac{1}{32\pi}\frac{\omega'}{m\omega(m+\omega(1-\cos\theta))}\mathrm{d}\cos\theta\,|\overline{\mathcal{M}}|^2\,.$$

In this expression, ω' is constrained by energy conservation to be a function of ω and θ. In fact,

$$\begin{aligned}
&{E_1'}^2 = (m+\omega-\omega')^2\\
&\quad\Leftrightarrow\quad \omega^2+\omega'^2+m^2-2\omega\omega'\cos\theta = m^2+\omega^2+\omega'^2+2\,m\omega-2\,\omega\omega'-2\,m\omega'\\
&\quad\Leftrightarrow\quad \omega'(m+\omega(1-\cos\theta)) = m\omega
\end{aligned}$$

which gives

$$\mathrm{d}\sigma = \frac{1}{32\pi}\frac{\omega'^2}{m^2\omega^2}\mathrm{d}\cos\theta\,|\overline{\mathcal{M}}|^2\,.$$

Finally using the expression for $|\overline{\mathcal{M}}|^2$ gives

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} &= \frac{1}{32\pi}\frac{\omega'^2}{m^2\omega^2}\,32\pi^2\alpha^2\left(\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+2\,m\,\frac{\omega'-\omega}{\omega\omega'}+m^2\left(\frac{\omega'-\omega}{\omega\omega'}\right)^2\right)\\
&= \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+\underbrace{2\,m\,\frac{\omega'-\omega}{\omega\omega'}+m^2\left(\frac{\omega'-\omega}{\omega\omega'}\right)^2+1}_{=\left(1+m\frac{\omega'-\omega}{\omega\omega'}\right)^2=\cos^2\theta}-1\right)\\
&= \frac{\pi\alpha^2}{m^2}\frac{\omega'^2}{\omega^2}\left(\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right)\,.
\end{aligned}$$

Problem 2: The Clifford algebra

1. Given a set of four matrices γ^μ which satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}\,,$$

show that the matrices $\gamma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ satisfy the Lorentz algebra:

$$[\gamma^{\kappa\lambda}, \gamma^{\rho\sigma}] = i\left(g^{\lambda\rho}\gamma^{\kappa\sigma} - g^{\kappa\rho}\gamma^{\lambda\sigma} - g^{\lambda\sigma}\gamma^{\kappa\rho} + g^{\kappa\sigma}\gamma^{\lambda\rho}\right)\,.$$

$$\begin{aligned}
[\gamma^{\kappa\lambda}, \gamma^{\rho\sigma}] &= -\frac{1}{16}\left[[\gamma^\kappa, \gamma^\lambda], [\gamma^\rho, \gamma^\sigma]\right] = -\frac{1}{16}\left[\gamma^\kappa\gamma^\lambda - \gamma^\lambda\gamma^\kappa, \gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho\right] \\
&= -\frac{1}{16}\left([\gamma^\kappa\gamma^\lambda, \gamma^\rho\gamma^\sigma] - [\gamma^\lambda\gamma^\kappa, \gamma^\rho\gamma^\sigma] - [\gamma^\kappa\gamma^\lambda, \gamma^\sigma\gamma^\rho] + [\gamma^\lambda\gamma^\kappa, \gamma^\sigma\gamma^\rho]\right) \\
&= -\frac{1}{16}\big(\gamma^\kappa\gamma^\lambda\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\kappa\gamma^\lambda - \gamma^\lambda\gamma^\kappa\gamma^\rho\gamma^\sigma + \gamma^\rho\gamma^\sigma\gamma^\lambda\gamma^\kappa \\
&\qquad - \gamma^\kappa\gamma^\lambda\gamma^\sigma\gamma^\rho + \gamma^\sigma\gamma^\rho\gamma^\kappa\gamma^\lambda + \gamma^\lambda\gamma^\kappa\gamma^\sigma\gamma^\rho - \gamma^\sigma\gamma^\rho\gamma^\lambda\gamma^\kappa\big)
\end{aligned}$$

Each of these eight terms is a product of four gamma matrices. We use the Clifford algebra for the two gamma matrices in the middle of each term (e.g. $\gamma^\lambda\gamma^\rho = 2g^{\lambda\rho} - \gamma^\rho\gamma^\lambda$ for the first term, $\gamma^\sigma\gamma^\kappa = 2g^{\sigma\kappa} - \gamma^\kappa\gamma^\sigma$ for the second one etc.:

$$\begin{aligned}
\ldots = &-\frac{1}{16}\big(2g^{\lambda\rho}[\gamma^\kappa, \gamma^\sigma] - 2g^{\kappa\rho}[\gamma^\lambda, \gamma^\sigma] - 2g^{\lambda\sigma}[\gamma^\kappa, \gamma^\rho] + 2g^{\kappa\sigma}[\gamma^\lambda, \gamma^\rho] \\
&\quad - \gamma^\kappa\gamma^\rho \underbrace{\gamma^\lambda\gamma^\sigma}_{2g^{\lambda\sigma} - \gamma^\sigma\gamma^\lambda} + \underbrace{\gamma^\rho\gamma^\kappa}_{2g^{\rho\kappa} - \gamma^\kappa\gamma^\rho} \gamma^\sigma\gamma^\lambda + \gamma^\lambda\gamma^\rho \underbrace{\gamma^\kappa\gamma^\sigma}_{2g^{\kappa\sigma} - \gamma^\sigma\gamma^\kappa} - \underbrace{\gamma^\rho\gamma^\lambda}_{2g^{\lambda\rho} - \gamma^\lambda\gamma^\rho} \gamma^\sigma\gamma^\kappa \\
&\quad + \underbrace{\gamma^\kappa\gamma^\sigma}_{2g^{\kappa\sigma} - \gamma^\sigma\gamma^\lambda} \gamma^\lambda\gamma^\rho - \gamma^\sigma\gamma^\kappa \underbrace{\gamma^\rho\gamma^\lambda}_{2g^{\rho\lambda} - \gamma^\lambda\gamma^\rho} - \gamma^\lambda\gamma^\sigma \underbrace{\gamma^\kappa\gamma^\rho}_{2g^{\kappa\rho} - \gamma^\rho\gamma^\kappa} + \underbrace{\gamma^\sigma\gamma^\lambda}_{2g^{\lambda\sigma} - \gamma^\lambda\gamma^\sigma} \gamma^\rho\gamma^\kappa\big) \\
= &-\frac{1}{16}\left(4g^{\lambda\rho}[\gamma^\kappa, \gamma^\sigma] - 4g^{\kappa\rho}[\gamma^\lambda, \gamma^\sigma] - 4g^{\lambda\sigma}[\gamma^\kappa, \gamma^\rho] + 4g^{\kappa\sigma}[\gamma^\lambda, \gamma^\rho]\right) \\
= &\; i\left(g^{\lambda\rho}\gamma^{\kappa\sigma} - g^{\kappa\rho}\gamma^{\lambda\sigma} - g^{\lambda\sigma}\gamma^{\kappa\rho} + g^{\kappa\sigma}\gamma^{\lambda\rho}\right)\,.
\end{aligned}$$

2. Verify that the Clifford algebra is satisfied by both the *Weyl representation* of γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}.$$

and the *Dirac-Pauli representation*

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

and find the unitary transformation that takes one into the other.

For either representation we have

$$\{\gamma^i, \gamma^j\} = \begin{pmatrix} -\{\sigma^i, \sigma^j\} & 0 \\ 0 & -\{\sigma^i, \sigma^j\} \end{pmatrix} = -2\begin{pmatrix} \delta^{ij} & 0 \\ 0 & \delta^{ij} \end{pmatrix} = 2g^{ij}\mathbb{1} \qquad (i, j = 1, 2, 3)\,.$$

Also, $(\gamma^0)^2 = \mathbb{1} = g^{00}\mathbb{1}$ is obvious. Moreover, in the Weyl representation,

$$\{\gamma^0, \gamma^i\} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} + \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} = 0$$

and in the Dirac-Pauli representation,

$$\{\gamma^0, \gamma^i\} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix} = 0\,.$$

With

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}$$

one has $U^\dagger U = \mathbb{1}$ and $U^\dagger \gamma^\mu_{\text{Weyl}} U = \gamma^\mu_{\text{D-P}}$.

3. Defining $\gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$, calculate

$$\{\gamma^5, \gamma^\mu\} \qquad \text{and} \qquad [\gamma^5, \gamma^{\mu\nu}]\,.$$

One has

$$\{\gamma^5, \gamma^0\} = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = 0$$

since in the first term, it takes three exchanges of γ matrices to move the γ^0 factor on the right all the way to the left, hence it picks up a factor $(-1)^3 = -1$. Similarly, for all the $\gamma^5\gamma^i$, it takes three exchanges to arrive at $\gamma^i\gamma^5$. Therefore,

$$\{\gamma^5, \gamma^\mu\} = 0\,.$$

Moreover,

$$\begin{aligned}
[\gamma^5, \gamma^{\mu\nu}] &= \frac{i}{4}\left[\gamma^5, [\gamma^\mu, \gamma^\nu]\right] = \frac{i}{4}\left([\gamma^5, \gamma^\mu\gamma^\nu] - [\gamma^5, 2\,g^{\mu\nu} - \gamma^\mu\gamma^\nu]\right) \\
&= \frac{i}{2}[\gamma^5, \gamma^\mu\gamma^\nu] = \frac{i}{2}\left(\{\gamma^\mu, \gamma^5\}\gamma^\nu - \gamma^\mu\{\gamma^5, \gamma^\nu\}\right) = 0\,.
\end{aligned}$$

Problem 3: The Dirac field

1. Show that

$$\left(\mathbb{1} + \frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}\right)\gamma^\mu \left(\mathbb{1} - \frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}\right) = \left(\mathbb{1} - \frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right)^\mu{}_\nu\gamma^\nu + \mathcal{O}(||\omega||^2)\,,$$

where the $M^{\rho\sigma}$ generate the vector representation of $\mathfrak{so}(1,3)$,

$$(M^{\kappa\lambda})_{\mu\nu} = i\left(\delta^\kappa{}_\mu\delta^\lambda{}_\nu - \delta^\kappa{}_\nu\delta^\lambda{}_\mu\right)\,.$$

Use this result to conclude that the Dirac Lagrangian

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$$

is invariant under proper orthochronous Lorentz transformations.

Expanding gives

$$\left(\mathbb{1} + \frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}\right)\gamma^\mu \left(\mathbb{1} - \frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}\right) = \gamma^\mu + \frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}\gamma^\mu - \frac{i}{2}\omega_{\rho\sigma}\gamma^\mu\gamma^{\rho\sigma} + \mathcal{O}(|\omega|^2)$$

Repeated use of the Clifford algebra leads to

$$[\gamma^\mu, \gamma^{\rho\sigma}] = i\left(g^{\mu\rho}\delta^\sigma_\nu - g^{\mu\sigma}\delta^\rho_\nu\right)\gamma^\nu = (M^{\rho\sigma})^\mu_\nu\,\gamma^\nu$$

and hence

$$\left(\mathbb{1} + \frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}\right)\gamma^{\mu}\left(\mathbb{1} - \frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}\right) = \gamma^{\mu} - \frac{i}{2}\omega_{\rho\sigma}\left(M^{\rho\sigma}\right)^{\mu}_{\ \nu}\gamma^{\nu} + \mathcal{O}(|\omega|^2)\,.$$

Exponentiating gives, in terms of the spinor Lorentz transformation $\tilde{\Lambda} = e^{-\frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}}$ and the vector Lorentz transformation $\Lambda = e^{-\frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma}}$,

$$\tilde{\Lambda}^{-1}\gamma\tilde{\Lambda} = \Lambda\gamma$$

where the transformation matrices on the LHS act only on spinor indices and the matrix on the RHS acts only on the vector index. Therefore the Dirac Lagrangian transforms as

$$\mathcal{L} \to \mathcal{L}' = \overline{\psi}\tilde{\Lambda}^{-1}\left(i\gamma^{\mu}(\Lambda^{-1}\partial)_{\mu} - m\right)\tilde{\Lambda}\psi = \overline{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi = \mathcal{L}.$$

2. Find the Euler-Lagrange equations obtained from the Dirac Lagrangian.

$$0 = -\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi)} + \frac{\partial\mathcal{L}}{\partial\psi} = -\partial_{\mu}\overline{\psi}i\gamma^{\mu} - m\overline{\psi}$$

$$0 = -\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\overline{\psi})} + \frac{\partial\mathcal{L}}{\partial\overline{\psi}} = (i\gamma^{\mu}\partial_{\mu} - m)\psi$$

These are the Dirac equation and its complex conjugate.