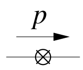
QFT, SOLUTIONS TO PROBLEM SHEET 8

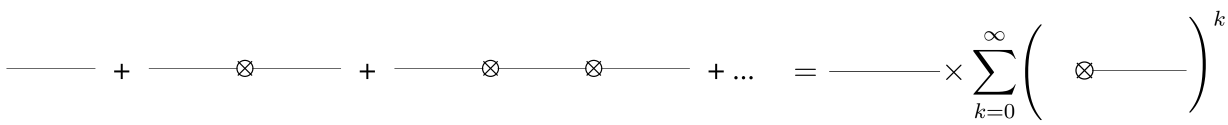
Problem 1: Counterterms in ϕ^4 theory

In ϕ^4 theory in renormalised perturbation theory, show that the Feynman rule for the "2-point vertex" associated to the counterterms δ_Z and δ_{m^2} is



$$= i(p^2\delta_Z - \delta_{m^2})\,.$$

We have



$$= \frac{i}{p^2 + m^2 - i\epsilon} \sum_{k=0}^{\infty} \left(\frac{-p^2\delta_Z + \delta_{m^2}}{p^2 - m^2 + i\epsilon} \right)^k$$

If this is to converge, one should obtain the usual limit of the geometric series, $\sum_n x^n = (1-x)^{-1}$, hence

$$\ldots = \frac{i}{p^2 - m^2 + i\epsilon} \frac{1}{1 - \frac{-p^2\delta_Z + \delta_{m^2}}{p^2 - m^2 + i\epsilon}} = \frac{i}{p^2 - m^2 + i\epsilon + p^2\delta_Z - \delta_{m^2}} = \frac{i}{Zp^2 - (m^2 + \delta_{m^2}) + i\epsilon}$$

Problem 2: Feynman parameters

Prove the following identities ($A \neq 0$ and $B \neq 0$ are real constants):

1.
$$\frac{1}{AB} = \int_0^1 \mathrm{d}x\, \frac{1}{(xA + (1-x)B)^2}$$

2.
$$\frac{1}{A^n B} = \int_0^1 \mathrm{d}x\, \frac{n x^{n-1}}{(xA + (1-x)B)^{n+1}}$$

3.
$$\frac{1}{A_1 \cdots A_n} = \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \delta(1 - x_1 - \ldots - x_n)\, \frac{(n-1)!}{(x_1 A_1 + \ldots + x_n A_n)^n}$$

1. For $A = B$ the identity is obvious. For $A \neq B$ one can directly integrate:

$$\int_0^1 \mathrm{d}x \frac{1}{(xA + (1-x)B)^2} = \frac{1}{B-A} \left[\frac{1}{xA + (1-x)B} \right]_{x=0}^{x=1} = \frac{1}{B-A} \left(\frac{1}{A} - \frac{1}{B} \right) = \frac{1}{AB}\,.$$

2. We differentiate m times on both sides of identity 1:

$$(-1)^m\, m!\, \frac{1}{A^{m+1}B} = \frac{\mathrm{d}^m}{dA^m}\frac{1}{AB} = \frac{\mathrm{d}^m}{dA^m}\int_0^1 \mathrm{d}x \frac{1}{(xA+(1-x)B)^2}$$
$$= \int_0^1 \mathrm{d}x \frac{\partial^m}{\partial A^m}\frac{1}{(xA+(1-x)B)^2} = \int_0^1 \mathrm{d}x \frac{(-1)^m\, x^m\,(m+1)!}{(xA+(1-x)B)^{2+m}}$$

The desired relation follows from dividing by $(-1)^m\, m!$ on both sides and setting $m = n-1$.

3. This is proved by induction. The statement is true for $n = 2$ (see part 1). Assuming it holds for some given n, let us show that it also holds for $n+1$:

$$\frac{1}{A_1 \ldots A_n A_{n+1}} \overset{\text{ass.}}{=} \int_0^1 \mathrm{d}y_1 \ldots \mathrm{d}y_n\, \delta(1-y_1-\ldots-y_n)\frac{(n-1)!}{(y_1A_1+\ldots+y_nA_n)^n}\frac{1}{A_{n+1}}$$
$$\overset{2.}{=} \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}y_1 \ldots \mathrm{d}y_n \delta(1-y_1-\ldots-y_n)\frac{n!z^{n-1}}{(zy_1A_1+\ldots+zy_nA_n+(1-z)A_{n+1})^{n+1}}$$

The integration domain can be explicitly parametrized as

$$\int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}y_1 \ldots \mathrm{d}y_n\, \delta(1-y_1-\ldots-y_n)$$
$$= \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}y_n \int_0^{1-y_n} \mathrm{d}y_{n-1} \int_0^{1-y_n-y_{n-1}} \mathrm{d}y_{n-2} \ldots \int_0^{1-y_n-y_{n-1}-\ldots-y_3} \mathrm{d}y_2 \Big|_{y_1 = 1-\sum_{k=2}^n y_k}$$

Now change variables: call $x_i = zy_i$ $(i = 1\ldots n)$ and $x_{n+1} = 1-z$, so that $\mathrm{d}y_i = \mathrm{d}x_i/z$ $(i = 2\ldots n)$ and $\mathrm{d}z = -\mathrm{d}x_{n+1}$:

$$\int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}y_1 \ldots \mathrm{d}y_n\, \delta(1-y_1-\ldots-y_n)$$
$$= \int_1^0 (-\mathrm{d}x_{n+1}) \int_0^z \frac{\mathrm{d}x_n}{z} \int_0^{z-x_n} \frac{\mathrm{d}x_{n-1}}{z} \cdots \int_0^{z-x_n-x_{n-1}-\ldots-x_3} \frac{\mathrm{d}x_2}{z}\Big|_{x_1 = z-\sum_{k=2}^n x_k}$$
$$= \int_0^1 \mathrm{d}x_{n+1} \int_0^{1-x_{n+1}} \frac{\mathrm{d}x_n}{z} \int_0^{1-x_{n+1}-x_n} \frac{\mathrm{d}x_{n-1}}{z} \cdots \int_0^{1-x_{n+1}-x_n-\ldots-x_3} \frac{\mathrm{d}x_2}{z}\Big|_{x_1 = z-\sum_{k=2}^n x_k}$$
$$= \int_0^1 \mathrm{d}x_1 \ldots \mathrm{d}x_{n+1}\, \delta(1-x_1-\ldots-x_{n+1}) z^{1-n}$$

One finally obtains

$$\frac{1}{A_1 \ldots A_n A_{n+1}} = \int_0^1 \mathrm{d}x_1 \ldots \mathrm{d}x_{n+1}\, \delta(1-x_1-\ldots x_{n+1})\, z^{1-n} \frac{n!z^{n-1}}{(x_1A_1+\ldots+x_{n+1}A_{n+1})^{n+1}}$$

as announced.