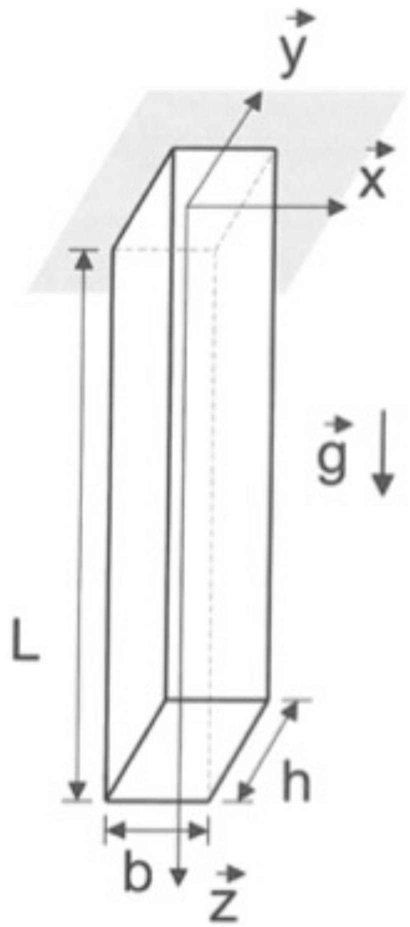
Feuille de TD 2-EF : Application de la méthode des Éléments Finis

Exercise 1: bar subjected to its own weight.

Consider a straight bar of length ***L***, with a constant cross-section ***S***, made of an isotropic homogeneous material of Young's modulus ***E*** and density ρ. The bar is fixed at one end and is subject to its own weight.



A] Analytical resolution.

- Establish the equilibrium equation, the behaviour law, the boundary conditions.
- Solve the problem analytically to obtain the displacement **u(x)** at any point.
- Calculate the normal force **N(x)** at any point.

B] Approximate resolution by the EF method.

1st case: the column is modeled by a linear element with two nodes.

- *Write the variational formulation of the problem.*
- *Recall the nodal unknowns of the bar element (i.e., under tension/compression) with linear shape functions. Express* $u(x)$ *as a function of these unknowns and the interpolation functions* $N_1(x)$ *and* $N_2(x)$*, which you will explicitly define.*
- *Formally express the problem in the form* $[K_e][Q] = [F_e]$*, where* $[K_e]$ *is the element stiffness matrix and* $[F_e]$ *is the element load vector.*
- *Explicitly compute the element vector* $[B_e]$*, and deduce* $[K_e]$*.*
- *Explicitly compute the element load vector* $[F_e]$*.*
- *Apply the boundary conditions and solve. Compare with the analytical solution and discuss.*

C] Approximate solution using the Finite Element (FE) method. Case 2: The column is modeled by three linear two-node elements of equal length $L/3$.

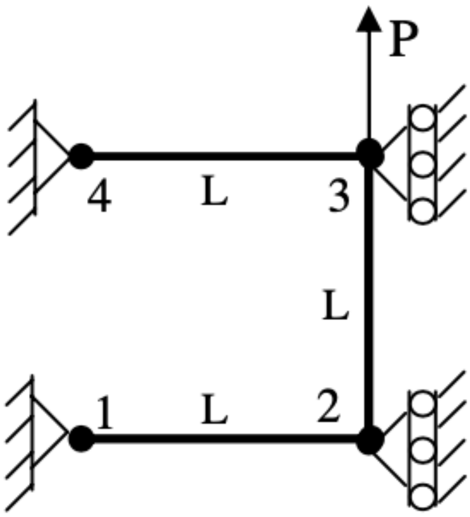
When using more than two elements, it becomes more convenient to work with the “reference element.”

- Provide a parameterization of the system. Once defined, it must remain unchanged.
- Recall the variable transformation used to switch from the x-coordinate system to the r-coordinate system.
- Recall the expressions of $\mathrm{N_1(r)}$ and $\mathrm{N_2(r)}$, the interpolation functions (fonction de fprme in French) in the reference coordinate system.
- Construct $[\mathrm{B_{ref}}]$ from $\mathrm{N_1(r)}$ and $\mathrm{N_2(r)}$. How is $[\mathrm{B_{ref}}]$ transformed into $[\mathrm{B_e}]$?
- Construct $[\mathrm{K_{ref}}]$. How is $[\mathrm{K_{ref}}]$ transformed into $[\mathrm{K_e}]$?
- Write the stiffness matrices $[\mathrm{K_e^{(1)}}]$, $[\mathrm{K_e^{(2)}}]$, and $[\mathrm{K_e^{(3)}}]$ for each element.
- Assemble the element matrices to form the global stiffness matrix $[\mathrm{K}]$.
- Construct the force vector $[\mathrm{F}]$.
- Apply the boundary conditions, solve the system $[\mathrm{K}][\mathrm{Q}] = [\mathrm{F}]$, compare the result with the analytical solution and with the one-element solution, and discuss.

D] Approximate solution using the Finite Element (FE) method. Case 3: The column is modeled by a quadratic three-node element.

- Provide a parameterization of the system. Once defined, it must remain unchanged.
- Recall the expressions of $\mathrm{N_1(r)}$, $\mathrm{N_2(r)}$, and $\mathrm{N_3(r)}$, the shape/interpolation functions in the reference coordinate system.
- Construct $[\mathrm{B_{ref}}]$.
- Construct $[\mathrm{K_{ref}}]$ and deduce $[\mathrm{K_e}]$.
- Construct $[\mathrm{F_{ref}}]$ and deduce $[\mathrm{F_e}]$.
- Apply the boundary conditions, solve the system $[\mathrm{K}][\mathrm{Q}] = [\mathrm{F}]$, and compare the result with the analytical solution, the one-element solution, and the two-element solution. Discuss.

Exercise 2: Truss case.
The objective of this exercise is to study a truss composed of only three bars. We will number the nodes and the elements as in the figure below.



- Write the boundary conditions of the problem. Specify the number of degrees of freedom (DOF) of the complete system.
- The stiffness matrix of a bar in the global coordinate system is given by:

$$[K]_e^g = \left(\frac{ES}{L}\right)_e \begin{bmatrix} cos^2(\theta) & cos(\theta)sin(\theta) & -cos^2(\theta) & -cos(\theta)sin(\theta) \\ cos(\theta)sin(\theta) & sin^2(\theta) & -cos(\theta)sin(\theta) & -sin^2(\theta) \\ -cos^2(\theta) & -cos(\theta)sin(\theta) & cos^2(\theta) & cos(\theta)sin(\theta) \\ -cos(\theta)sin(\theta) & -sin^2(\theta) & cos(\theta)sin(\theta) & sin^2(\theta) \end{bmatrix}$$

Course question: How do we go from the expression of $[K_e]$ (the stiffness matrix in the local coordinate system of the bar) to $[K_{eg}]$, the stiffness matrix in the global coordinate system?

- Compute the stiffness matrices for each bar.
- Assemble the matrices and construct the global load vector $[F_g]$.
- Determine the nodal displacements.

Exercise 3: For practice.
Determine the nodal displacements for the following truss: